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Equations différentielles stochastiques de type McKean-Vlasov et leur contrôle optimal

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Dedicace

- *A mes chers Parents pour tout ce qu'ils m'ont donné.*
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Resume

Nous considérons les équations différentielles stochastiques (EDS) de Mc Kean-Vlasov, qui sont des EDS dont les coefficients de dérive et de diffusion dépendent non seulement de l'état du processus inconnu, mais également de sa loi de probabilité. Ces EDS, également appelées EDS à champ moyen, ont d'abord été étudiées en physique statistique et représentent en quelque sorte le comportement moyen d'un nombre infini de particules. Récemment, ce type d'équations a suscité un regain d'intérêt dans le contexte de la théorie des jeux à champ moyen. Cette théorie a été inventée par P.L. Lions et J.M. Lasry en 2006, pour résoudre le problème de l'existence d'un équilibre de Nash approximatif pour les jeux différentiels, avec un grand nombre de joueurs. Ces équations ont trouvé des applications dans divers domaines tels que la théorie des jeux, la finance mathématique, les réseaux de communication et la gestion des ressources pétrolières. Dans cette thèse, nous avons étudié les questions de stabilité par rapport aux données initiales, aux coefficients et aux processus directeurs des équations de McKean-Vlasov. Les propriétés génériques de ce type d' équations stochastiques, telles que l'existence et l'unicité, la stabilité par rapport aux paramètres, ont été examinées. En théorie du contrôle, notre attention s'est portée sur l'existence et l'approximation de contrôles relaxés pour les systèmes gouvernés par des EDS de Mc Kean-Vlasov.

Key words : EDS de Mc Kean-Vlasov – EDS de type champ moyen – Stabilité – Approximation - Propriété générique – Martingale – Existence – Control de type champ moyen – Jeu à champ moyen - Contrôle relaxé - Contrôle strict.

Abstract

We consider Mc Kean-Vlasov stochastic differential equations (SDEs), which are SDEs where the drift and diffusion coefficients depend not only on the state of the unknown process but also on its probability distribution. These SDEs called also mean- field SDEs were first studied in statistical physics and represent in some sense the average behavior of an infinite number of particles. Recently there has been a renewed interest for this kind of equations in the context of mean-field game theory. Since the pioneering papers by P.L. Lions and J.M. Lasry, mean-field games and mean-field control theory has raised a lot of interest, motivated by applications to various fields such as game theory, mathematical finance, communications networks and management of oil resources. In this thesis, we studied questions of stability with respect to initial data, coefficients and driving processes of Mc Kean-Vlasov equations. Generic properties for this type of SDEs, such as existence and uniqueness, stability with respect to parameters, have been investigated. In control theory, our attention were focused on existence, approximation of relaxed controls for controlled Mc Kean-Vlasov SDEs.

Key words : Mc Kean-Vlasov SDE – Mean-field SDE – Stability – Approximation - Generic property – Stability – Martingale – Existence – Mean-field control – Mean-field game - Relaxed control - Strict control.

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Résumé de la thèse

Le cadre général de cette thèse est l'analyse stochastique et le contrôle optimal des systèmes régis par des équations différentielles stochastiques. Ce sont des systèmes dynamiques évoluant de façon aléatoire dans le temps, ce qui pourrait être considéré comme une généralisation des équations différentielles ordinaires.Le principal outil du calcul stochastique est le calcul Itô, qui a été introduit par K. Itô, dans les années 40 pour construire les trajectoires des processus de Markov associés à des opérateurs elliptiques. Pour ce faire, il a inventé le calcul stochastique qui porte son nom, pour donner un sens aux intégrales stochastiques et aux équations différentielles stochastiques. Le calcul stochastique a trouvé de nombreuses applications telles que la théorie des EDPs, l'ingénierie, la biologie et l'économie. Plus récemment, l'application la plus spectaculaire est son rôle dans la finance mathématique. Robert Merton a introduit le calcul stochastique dans l'étude de la finance et, en même temps, Fischer Black et Myron Scholes ont développé leur formule de tarification des options. De nos jours, la finance mathématique est devenue l'une des sciences interdisciplinaires à la croissance la plus rapide. Elle s'appuie sur les disciplines de la théorie des probabilités, des statistiques, du calcul scientifique et des équations aux dérivées partielles, pour fournir des modèles et établir des relations entre les prix des actifs, les mouvements du marché et les taux d'intérêt. Dans cette thèse, notre attention sera concentrée sur une classe spéciale d'équations différentielles stochastiques, appelées équations différentielles stochastiques de McKean-Vlasov (MVSDE) ou équations différentielles stochastiques de type champ moyen (MFSDE).

Ce sont des EDSs est donnée comme suit

$$\begin{cases} dX_t = b(t, X_t, P_{X_t})dt + \sigma(t, X_t, P_{X_t})dW_t \\ X_0 = x. \end{cases}$$

où b est la dérive et σ est le coefficient de diffusion et (W_t) est un mouvement brownien. Notez que dans les MFSDEs, les coefficients dépendent non seulement de la variable d'état mais également de sa distribution marginale. Cela apporte une difficulté supplémentaire par rapport aux SDE Itô classiques. Les solutions d'une telle équation sont connues dans la littérature sous le nom de diffusions non linéaires.

Les MFSDE ont d'abord été étudiées en physique statistique par M. Kac [46], en tant que contre partie stochastique de l'équation de Vlasov du plasma [68]. L'étude d'une telle équation a été réalisée par McKean [56], voir aussi Snitzman [66] pour une excellente introduction à ce domaine. Ces équations ont été obtenues en tant que limites de certains systèmes de particules à interaction faible, lorque le nombre de particules tend vers l'infini. Ce type de résultat d'approximation est appelé "propagation du chaos", qui dit que lorsque le nombre de particules tend vers l'infini, les équations définissant l'évolution des particules pourraient être remplacées par une seule équation, appelée équation de McKean-Vlasov. Cette équation de champ moyen, représente en quelque sorte le comportement moyen du nombre infini de particules, voir [45, 66] pour plus de détails.

Récemment, il y a eu un regain d'intérêt pour les MVSDEs, dans le contexte de la théorie des jeux à champ moyen (MFG). Cette théorie a été inventée par P.L. Lions et J.M. Lasry en 2006 [51], pour résoudre le problème de l'existence d'un équilibre Nash approximatif pour les jeux différentiels, avec un grand nombre de joueurs. Depuis les articles précédents [43, 51], la théorie des jeux à champ moyen a suscité beaucoup d'intérêt, motivée par des applications dans divers domaines tels que la théorie des jeux, la finance mathématique, les réseaux de communication et la gestion des ressources pétrolières.

Le contrôle des EDSs de Mc Kean Vlasov est un domaine relativement nouveau, dans la théorie du contrôle stochastique. Il est essentiellement motivé par le comportement de grandes populations en interaction et trouve des applications en économie et en finance notamment l'étude du risque systémique. D'autres applications se sont avérées importantes dans les réseaux sociaux, la physique (physique statistique) et la biologie. En particulier, le problème bien connu de sélection de portefeuille du problème de variance moyenne de Markowitz est un exemple typique du problème de contrôle de Mc Kean Vlasov où l'on doit minimiser une fonction objective impliquant une fonction quadratique, en raison du terme de variance. Le principal inconvénient, lorsqu'il s'agit de tels problèmes de contrôle stochastique à champ moyen, est que le principe d'optimalité de Bellmann ne tient pas. Pour ce type de problèmes, le principe du maximum stochastique fournit un outil puissant pour les résoudre. On peut également se référer à la référence la plus récente sur le sujet [20] et à la liste complète des références contenue.

Cette thèse contient quatre articles, traitant de diverses questions d'existence, d'unicité et de stabilité des MVSDEs ainsi que du contrôle optimal de telles équations. Dans la suite, nous décrivons en détail le contenu de ces articles.

Dans le premier article, nous étudions certaines propriétés des MVSDE telles que les propriétés d'existence, d'unicité et de stabilité. En particulier, nous prouvons un théorème d'existence et d'unicité pour une classe de MVSDE sous une condition de type Osgood sur les coefficients, améliorant le cas connu de Lipschitz. De plus, nous étudions la stabilité par rapport aux conditions initiales, aux coefficients et aux processus directeurs, qui sont des martingales continues et des processus de variation bornés. Ces propriétés seront étudiées sous la condition de Lipschitz en ce qui concerne la variable d'état et la distribution et généraliser les propriétés connues pour les EDSs d'Itô classiques. De plus, nous prouvons que dans le contexte du contrôle stochastique de systèmes gouvernés par des MVSDE, les problèmes de contrôle relaxé et strict ont la même fonction de valeur. L'idée est d'injecter l'espace des contrôles stricts dans l'ensemble des contrôles de valeur à valeurs mesure, appelés contrôles relaxés, qui jouit de bonnes propriétés de compacité. Sous la condition de Lipschitz, nous prouvons que les fonctions de valeur sont égales. Notons que ce résultat généralise [6, 7] établis pour une classe spéciale de MVSDE, où la dépendance du coefficient sur la variable de distribution se fait via une forme linéaire de la distribution.

Dans le deuxième article, notre attention sera concentrée sur les propriétés de stabilité forte de la solution d'une MVSDE sous l'hypothèse d'unicité trajectorielle des solutions et de la continuité des coefficients. Comme les coefficients sont continus mais sans régularité supplémentaire, on ne peut pas s'attendre à appliquer le lemme de Gronwall. Au lieu du lemme de Gronwall, nous utilisons des arguments de relative compacité des lois et le célèbre théorème de sélection de Skorokhod pour prouver les résultats de convergence souhaités. En particulier, nous montrons que le schéma polygonal d'Euler est convergent à condition sous l'unicité trajectorielle. Cela nous fournit un moyen efficace de construire des solutions fortes des MVSDE. De plus, nous prouvons que la solution est stable par rapport à une petite perturbation de la condition initiale et des coefficients. Nos résultats généralisent les résultats similaires prouvés pour les EDSs de Itô classiques [9, 37, 47]. De plus, nous montrons que l'ensemble des coefficients bornés uniformément continus pour lesquels l'existence et unicité fortes sont vérifiées est une propriété générique au sens de Baire. Cela signifie qu'au sens de la catégorie de Baire, la plupart des MVSDEs avec des coefficients bornés uniformément continus ont des solutions uniques. Ce dernier résultat étend notamment les résultats de [2, 9, 10] aux MVSDEs.

Le troisième article est consacré à la convergence du schéma numérique de Carathéodory pour une classe de MVSDEs non linéaires. Ce schéma d'approximation est défini par une suite de solutions de MVSDEs avec de petits retards. Nous commençons par prouver sous les conditions de Lipschitz, que le schéma converge vers la solution unique de notre éqiuation. La preuve est basée sur des estimations des solutions et les arguments habituels du calcul stochastique. Ce résultat étend ceux de [**?**, **?**] aux MVSDE. Dans notre deuxième résultat principal, nous prouvons le résultat de convergence, quelle que soit la condition sur les coefficients, garantissant l'unicité trajectorielle, à condition que les coefficients soient continus et satisfassent une certaine condition de croissance linéaire. La preuve est basée sur la tensionde lois des processus considérés et le théorème de Skorokhod. Nous utilisons un résultat profond caractérisant la convergence en probabilité en termes de faible convergence de couples de sous-suites. Contrairement au schéma d'approximations successives de Picard, qui est valide sous des hypothèses plutôt fortes sur les coefficients, le schéma de Carathéodory converge dans toutes les conditions garantissant l'unicité forte.

Le quatrième article est consacré à l'étude de l'existence, ainsi que certaines propriétés d'approximation des contrôles optimaux. Notre point de départ est un problème de contrôle (appelé problème de contrôle strict), qui n'admet pas nécessairement un contrôle optimal. Nous construisons un deuxième problème de contrôle, appelé problème de contrôle relaxé, avec deux propriétés principales. Le premier est que le problème relaxé admet une solution optimale. La deuxième propriété est que les contrôles relaxés ainsi que leurs états et leurs fonctions de coût pourraient être approximés au moyen de contrôles stricts ainsi de leurs états et leurs fonctions de coût. Cette dernière propriété de stabilité est utile dans les problèmes numériques et d'ingénierie. En effet, il est plus pratique de manipuler des contrôles presque optimaux qui sont des fonctions au lieu de contrôles optimaux qui sont des processus à valeurs mesures. Pour atteindre cet objectif, nous appliquons la méthode dite de compactification pour montrer l'existence d'un contrôle relaxé optimal. Cette méthode est basée sur la compacité relative des lois des processus considérés, qui ne nécessite aucune régularité des coefficients ou de la fonction valeur. Nous prouvons deux résultats principaux. Le premier est un résultat d'approximation forte du problème de contrôle relaxé par une suite de problèmes de contrôle stricts. Cela signifie en particulier que la relaxation du problème initial n'affecte pas la fonction de valeur de notre problème. Ceci est valide avec des coefficients continus et l'unicité forte des solutions de l'équation d'état. Ce résultat pourrait être considéré comme un résultat de stabilité par rapport à la variable de contrôle et pourrait être utilisé pour effectuer des approximations numériques. Le deuxième résultat principal est l'existence d'un contrôle relaxé optimal sous la condition de continuité des coefficients. Notons que nos résultats améliorent le résultat l'existence connu pour les systèmes gouvernés par des EDSs de Itô [5, 9] ainsi que [6, 7]. Les principaux ingrédients utilisés dans les preuves sont les critères de tension des lois et le théorème de sélection de Skorokhod.

0.1 Stabilité des équations différentielles stochastiques de type McKean-Vlasov et applications

0.1.1 Introduction

Notre principal objectif dans cette partie est d'étudier certaines propriétés des MVSDEs telles que les propriétés d'existence, d'unicité et de stabilité. En particulier, nous prouvons un théorème d'existence et d'unicité pour une classe de MVSDE sous une condition de type Osgood sur les coefficients, améliorant le cas Lipschitzien. Il est bien connu que les propriétés de stabilité des systèmes dynamiques déterministes ou stochastiques sont cruciales dans l'étude de tels systèmes. Cela signifie que les trajectoires ne changent pas trop sous de petites perturbations. Nous étudions la stabilité par rapport aux conditions initiales, aux coefficients et aux processus directeurs, qui sont des martingales continues et des processus de variation bornée. Ces propriétés seront étudiées sous la condition de Lipschitz, en ce qui concerne la variable d'état et la loi et généraliser les propriétés connues pour les EDSs d'Itô classiques [9, 44]. De plus, nous prouvons que dans le contexte du contrôle stochastique de systèmes guvernés par des MVSDEs, les problèmes de contrôle relaxé et strict ont la même fonction de valeur. Comme il est bien connu que si la condition de convexité de type Filipov n'est pas remplie, il n'y a aucun moyen de prouver l'existence d'un contrôle strict. L'idée est alors d'injecter les contrôles stricts dans l'ensemble de contrôles à valeurs mesures, appelés contrôles relaxés, qui jouit de bonnes propriétés de compacité. Ainsi, pour que le problème de contrôle relaxé soit une véritable extension du problème initial, les fonctions de valeur des deux problèmes doivent être les mêmes. Sous la condition de Lipschitz, nous prouvons que les fonctions de valeur sont égales. Notez que ce résultat generalise aux équations générales de McKean Vlasov ceux dans [6, 7] établis pour une classe spéciale de MVSDEs.

0.1.2 Formulation du problème

Soit $(\Omega, \mathcal{F}, P)$ un espace de probabilité, équipé d'une filtration $(\mathcal{F}_t)$, satisfaisant aux conditions habituelles et (B_t) un $(\mathcal{F}_t, P)$ -Mouvement brownien d-dimensionnel . Considérons l'équation différentielle stochastique de McKean-Vlasov suivante également appelée équation différentielle stochastique à champ moyen (MVSDE)

$$\begin{cases} dX_t = b(t, X_t, \mathbb{P}_{X_t})ds + \sigma(t, X_t, \mathbb{P}_{X_t})dB_s \\ X_0 = x \end{cases} \tag{1}$$

Notons que pour ce type de SDE, la dérive b et le coefficient de diffusion σ dépendent non seulement de la position, mais aussi de la distribution marginale de la solution.

Les hypothèses suivantes seront valides tout au long de cet article.

Notons $\mathcal{P}_2(\mathbb{R}^d)$ l'espace des mesures de probabilité à moment fini du second ordre. Soit pour chaque $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ $\int |x|^2 \mu(dx) < +\infty$.

$(\mathbf{H_1})$ On suppose que

$$b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d$$
$$\sigma : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$$

sont des fonctionns Boreliennes mesurables et il existe $C > 0$ tel que poutr tout $(t, x, \mu) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$:

$$|b(t, x, \mu)| + |\sigma(t, x, \mu)| \leq C\,(1 + |x|)$$

$(\mathbf{H_2})$ Il existe $L > 0$ tel que pour tout $t \in [0,T]$, $x,\ x' \in \mathbb{R}^d$ et $\mu,\ \mu' \in \mathcal{P}_2(\mathbb{R}^d)$,

$$|b(t, x, \mu) - b(t, x', \mu')| \leq L[|x - x'| + W_2(\mu, \mu')]$$
$$|\sigma(t, x, \mu) - \sigma(t, x', \mu')| \leq L[|x - x'| + W_2(\mu, \mu')]$$

où W_2 est la métrique de Wasserstein.

0.1.3 Existence et unicité des solutions

L'unicité sous l'hypothèse d'Osgood

Dans cette section, nous relaxons la condition globale de Lipschitz en la variable d'état. Nous prouverons l'existence et l'unicité d'une solution lorsque les coefficients sont globalement

Lipschitz dans la variable de distribution et satisfont une condition de type Osgood en la variable d'état. Pour être plus précis, considérons l'équation suivante

$$\begin{cases} dX_t = b(t, X_t, \mathbb{P}_{X_t})ds + \sigma(t, X_t)dB_s \\ X_0 = x \end{cases} \tag{2}$$

Supposons que b et σ sont des fonctions Boreliennes bornées à valeurs réelles satisfaisant :

$(\mathbf{H_4})$ Il existe $C > 0$, tel que pour tout $x \in \mathbb{R}$ and $(\mu, \nu) \in \mathcal{P}_1(\mathbb{R}) \times \mathcal{P}_1(\mathbb{R})$:

$$|b(t, x, \mu) - b(t, x, \nu)| \leq CW_1(\mu, \nu)$$

$(\mathbf{H_5})$ Il existe une fonction strictement croissante $\rho(u)$ sur $[0, +\infty)$ telle que $\rho(0) = 0$ and ρ^2 est convexe satisfaisant $\int\limits_{0^+} \rho^{-2}(u)du = +\infty$, telle que pour tous $(x, y) \in \mathbb{R} \times \mathbb{R}$ and $\mu \in \mathcal{P}_2(\mathbb{R})$, $|\sigma(t, x) - \sigma(t, y)| \leq \rho(|x - y|)$.

$(\mathbf{H_6})$ Il existe une fonction strictement croissante $\kappa(u)$ sur $[0, +\infty)$ telle que $\kappa(0) = 0$ and κ est concave satisfaisant $\int\limits_{0^+} \kappa^{-1}(u)du = +\infty$,telle que pour tous $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R})$, $|b(t, x, \mu) - b(t, y, \mu)| \leq \kappa(|x - y|)$.

Dans le théorème suivant, nous démontrerons l'unicité forte pour l'équation (2)sous une condition de type Osgood en la variable d'état. Ce résultat améliore le Théorème 3.2 dans [44], établi pour les EDS classiques d'Itô et le Théorème 4.21 dans [20], au moins pour les MVSDEs avec un coefficient de diffusion ne dépendant pas de la loi du processus.

Theorem 0.1 *Sous les hypothèses* $(\mathbf{H_4}) - (\mathbf{H_6})$, *l'équation (2) possède la propriété d'uncité forte.*

Remark. La continuité et la bornitude des coefficients impliquent l'existence d'une solution faible *(see [45] Proposition 1.10).* Ensuite, par le théorème bien connu de Yamada - Watanabe appliqué à l'équation (2) (voir *[48]* exemple 2.14, page 10), l'unicité forte prouvée dans le dernier théorème implique l'existence et l'unicité d'une solution forte.

0.1.4 Convergence de l'approximation successive de Picard

Supposons que $b(t,x,\mu)$ et $\sigma(t,x,\mu)$ satisfont les hypothèses $(\mathbf{H_1})$, $(\mathbf{H_2})$. Nous allons prouver la convergence du schéma d'itération de Picard. Ce schéma est utile pour les calculs numériques de la solution unique de l'équation (1).Soit $(X_t^0) = x$ pour tout $t \in [0,T]$ on définit $\left(X_t^{n+1}\right)$ comme solution de l'EDS suivante

$$\begin{cases} dX_t^{n+1} = b(t, X_t^n, \mathbb{P}_{X_t^n})dt + \sigma(t, X_t^n, \mathbb{P}_{X_t^n})dB_t \\ X_0^{n+1} = x \end{cases}$$

Theorem 0.2 *Sous les hypothèses* $(\mathbf{H_1})$, $(\mathbf{H_2})$, *la suite* (X^n) *converge vers* X *la solution unique de (1)*

$$E[\sup_{t\leq T} |X_t^n - X_t|^2] \to 0$$

0.1.5 Stabilité par rapport à la condition initiale

Dans cette section, nous étudierons la stabilité des MVSDEs par rapport de petites perturbations de la condition initiale.

On note par (X_t^x) la solution unique de (4.1) tel que $X_0^x = x$

$$\begin{cases} dX_t^x = b(t, X_t^x, \mathbb{P}_{X_t^x})dt + \sigma(t, X_t^x, \mathbb{P}_{X_t^x})dB_t \\ X_0^x = x \end{cases}$$

Theorem 0.3 *Supposons que* $b(t,x,\mu)$ *et* $\sigma(t,x,\mu)$ *satisfont* $(\mathbf{H_1})$, $(\mathbf{H_2})$, *alors l'application*

$$\Phi : \mathbb{R}^d \longrightarrow L^2(\Omega, \mathcal{C}([0,T], \mathbb{R}^d))$$

définie par $(\Phi(x)_t) = (X_t^x)$ est continue.

0.1.6 Stabilité par rapport aux coefficients

Dans cette section, nous établirons la stabilité des MVSDEs par rapport a des perturbations faibles des cofficients b et ■. Considérons des suites de fonctions (b_n) et (σ_n) et considérons l'équation MVSDE :

$$dX_t^n = b_n(t, X_t^n, \mathbb{P}_{X_t^n})dt + \sigma_n(t, X_t^n, \mathbb{P}_{X_t^n})dB_t \tag{3}$$
$$X_0^n = x$$

Le théorème suivant nous donne la dépendance continue de la solution par rapport aux coefficients.

Theorem 0.4 *Supposons que les fonctions* $b(t, x, \mu)$, $b_n(t, x, \mu)$, $\sigma(t, x, \mu)$ *et* $\sigma_n(t, x, \mu)$ *satisfont* $(\mathbf{H_1})$, $(\mathbf{H_2})$. *Supposons en outre que pour chaque* $T > 0$ *et chaque ensemble compact* K, *il existe* $C > 0$ *tel que*

$i)$ $\sup_{t\leq T}(|b_n(t, x, \mu)| + |\sigma_n(t, x, \mu)|) \leq C(1 + |x|)$,
$ii)$ $\lim_{n\to\infty} \sup_{t\leq T} \sup_{x\in K} \sup_{\mu\in\mathcal{P}_2(\mathbb{R}^d)} ||b_n(t, x, \mu) - b(t, x, \mu)|| + ||\sigma_n(t, x, \mu) - \sigma(t, x, \mu)|| = 0$
Alors

$$\lim_{n\to\infty} E\left[\sup_{t\leq T} |X_t^n - X_t|^2\right] = 0$$

où (X_t^n) et (X_t) sont respectivement des solutions de (3) et (1).

0.1.7 Stabilité par rapport a des processus directeurs

Dans cette section, nous considérons les MVSDEs dirigées par des semi-martingales continues. Soit $b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ et $\sigma : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d\times d}$ des fonctions continues bornées.

Nous considérons les MVSDEs dirigées par des semi-martingales continues de la forme suivante

$$\begin{cases} dX_t = b(t, X_t, \mathbb{P}_{X_t})dA_t + \sigma(t, X_t, \mathbb{P}_{X_t})dM_t \\ X_0 = x \end{cases} \tag{4}$$

où A_t est un processus continu adapté à variation bornée et M_t est une martingale locale continue.

Considérons la suite suivante de MVSDEs

$$\begin{cases} dX_t^n = b(t, X_t^n, \mathbb{P}_{X_t^n})dA_t^n + \sigma(t, X_t^n, \mathbb{P}_{X_t^n})dM_t^n \\ X_0^n = x \end{cases} \tag{5}$$

où (A^n) est une suite de processus continus $\mathcal{F}_t$-adaptés à variation bornée et M^n des $(\mathcal{F}_t, \mathbb{P})$-local martingales continues.

Supposons que (A, A^n, M, M^n) satisfait à :

$(\mathbf{H_7})$

1) La famille (A, A^n, M, M^n) est bornée dans $\mathbb{C}([0,1])^4$.

2) $(M^n - M)$ converge vers 0 en probabilité dans $\mathbb{C}([0,1])$ quand n tends to $+\infty$.

3) La variation totale $(A^n - A)$ converge vers 0 en probabilité quand n tend vers $+\infty$.

Theorem 0.5 *Supposons que $b(t,x,\mu)$ et $\sigma(t,x,\mu)$ satisfont $(\mathbf{H_1})$, $(\mathbf{H_2})$. Supposons en outre que (A, A^n, M, M^n) vérifie $(\mathbf{H_7})$, Alors*

$$\lim_{n\to\infty} E[\sup_{t\le T} |X_t^n - X_t|^2] = 0$$

où (X_t^n) et (X_t) sont respectivement des solutions de (5) et (4).

0.1.8 Approximation des problèmes de contrôle relaxé

Soit $\mathbb{A}$ un espace métrique compact appelé l'espace d'actions. Un contrôle strict (u_t) est un processus mesurable, $\mathcal{F}_t-$ adapté avec des valeurs dans l'espace d'actions $\mathbb{A}$. Nous notons $\mathcal{U}_{ad}$ l'espace de contrôles stricts.

Le processus d'état correspondant à un contrôle strict est la solution unique de l'équation suivante

$$\begin{cases} dX_t = b(t, X_t, \mathbb{P}_{X_t}, u_t)ds + \sigma(t, X_t, \mathbb{P}_{X_t}, u_t)dB_s \\ X_0 = x \end{cases} \tag{6}$$

et la fonction de coût correspondante est donnée par

$$J(u) = E\left[\int_0^T h(t, X_t, \mathbb{P}_{X_t}, u_t dt + g(X_T, \mathbb{P}_{X_T})\right].$$

Le problème est de minimiser $J(u)$ sur l'espace $\mathcal{U}_{ad}$ de contrôles stricts et de trouver $u^* \in \mathcal{U}_{ad}$ tel que $J(u^*) = \inf\{J(u), u \in \mathcal{U}_{ad}\}$.

Considérons les hypothèses suivantes dans cette section.

$(\mathbf{H_4})$ $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{A} \longrightarrow \mathbb{R}^d$, $\sigma : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{A} \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$, sont continues bornées. .

$(\mathbf{H_5})$ $b(t,.,.,a)$ et $\sigma(t,.,.,a)$ sont Lipschitz uniformément en $(t,a) \in [0,T] \times \mathbb{A}$.

$(\mathbf{H_6})$ $h : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{A} \longrightarrow \mathbb{R}$ et $g : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, sont des fonctions continues bornées, telles que $h(t,.,.,a)$ est Lipschitz en (x,μ).

On note $\mathbb{V}$ l'ensemble des mesures de produit μ sur $[0,T] \times \mathbb{A}$ dont projection sur $[0,T]$ coïncide avec la mesure de Lebesgue dt. $\mathbb{V}$ en tant que sous-espace fermé de l'espace des mesures positives de radon $\mathbb{M}_+([0,T] \times \mathbb{A})$ est compact pour la topologie de convergence faible.

Definition 0.1 *Un contrôle relaxé sur l'espace de probabilité filtré $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ est une variable aléatoire $\mu = dt.\mu_t(da)$ avec des valeurs dans $\mathbb{V}$, tel que $\mu_t(da)$ est progressivement mesurable par rapport à $(\mathcal{F}_t)$ et tel que pour chaque t, $1_{(0,t]}.\mu$ est $\mathcal{F}_t-$mesurable.*

Remark 0.1 *L'ensemble $\mathcal{U}_{ad}$ des contrôles stricts est injecté dans l'ensemble des contrôles relaxés en identifiant u_t avec $dt\delta_{u_t}(da)$.*

Il a été prouvé dans [29] pour les problèmes de contrôle classiques et dans [7] que le processus d'état relaxé correspondant à un contrôle relaxé doit satisfaire une MVSDE dirigée par une mesure martingale au lieu d'un mouvement brownien. C'est à dire le processus d'état relaxé satisfait à

$$\begin{cases} dX_t = \int_{\mathbb{A}} b(t, X_t, \mathbb{P}_{X_t}, a)\, \mu_t(da)dt + \int_{\mathbb{A}} \sigma(t, X_t, \mathbb{P}_{X_t}, a)\, M(da, dt) \\ X_0 = x, \end{cases} \tag{7}$$

où M est une mesure martingale orthogonale continue, avec intensité $dt\mu_t(da)$.En utilisant les mêmes outils que dans le théorème 3.1, il n'est pas difficile de prouver que (7) admet une solution forte unique. Le lemme suivant, connu dans la littérature sur les contrôles sous le nom de Chattering Lemma, indique que l'ensemble de contrôles stricts est un sous-ensemble dense dans l'ensemble de contrôles relaxés.

Lemma 0.1 *i) Soit (μ_t) un contrôle relaxé.Alors, il existe une suite de processus adaptés (u_t^n) avec des valeurs dans $\mathbb{A}$, telle que la suite de mesures aléatoires $\left(\delta_{u_t^n}(da)\,dt\right)$ converge dans $\mathbb{V}$ vers $\mu_t(da)\,dt,\ P-a.s.$*
ii) Pour toute g continue dans $[0,T]\times\mathbb{M}_1(\mathbb{A})$ tel que $g(t,.)$ est linéaire, on a $\lim\limits_{n\to+\infty}\int_0^t g(s,\delta_{u_s^n})ds=\int_0^t g(s,\mu_s)ds$ uniformément pour toute $t\in[0,T]\,,\ P-a.s.$

Soit X_t^n la solution de l'équation d'état *(*6) corresponding to u^n, où u^n est un contrôle strict défini dans le dernier lemme. Si on note $M^n(t,F)=\int_0^t\int_F\delta_{u_s^n}(da)dW_s$, alors $M^n(t,F)$ est une mesure martingale orthogonale et X_t^n peut être écrit sous une forme relaxée comme suit

$$\begin{cases} dX_t^n=\int\limits_{\mathbb{A}} b(t,X_t^n,\mathbb{P}_{X_t^n},a)\delta_{u_t^n}(da)dt+\int\limits_{\mathbb{A}}\sigma(t,X_t,\mathbb{P}_{X_t^n},a)M^n(dt,da) \\ X_0=x \end{cases}$$

La proposition suivante donne la continuité de la dynamique *(*7) par rapport à la variable de contrôle.

Proposition 0.1 *i) Si X_t, X_t^n désignent les solutions de l'équation (7) correspond à μ et μ^n, alors pour chaque $t\le T$, $\lim\limits_{n\to+\infty}E(|X_t^n-X_t|^2)=0.$*
ii) Soient $J(u^n)$ et $J(\mu)$ les coûts attendus correspondant respectivement à u^n et μ, alors $(J\,(u^n))$ converge vers $J\,(\mu)\,.$

Remark 0.2 *Selon la dernière proposition, il est clair que l'infimum parmi les controles relaxés est égal à l'infimum parmi les contoles strictes, ce qui implique que les fonctions de valeur pour les modèles relaxés et strict sont les mêmes.*

0.2 Stabilité et généricité des équations différentielles stochastiques de type McKean-Vlasov à coefficients continus

0.2.1 Introduction

Notre objectif dans cette partie est d'étudier les propriétés de stabilité forte de la solution de (8) sous la condition d'unicité forte de la solution et la continuité des coefficients. Comme les coefficients ne sont que continus sans régularité supplémentaire, on ne peut pas s'attendre à appliquer le lemme de Gronwall. A la place, nous utiliserons des arguments de tension des lois et le célèbre théorème de sélection de Skorokhod pour prouver les résultats de convergence souhaités. Bien sûr, il ne faut pas s'attendre à une vitesse de convergence précise car cette dernière propriété est basée sur la régularité des coefficients.

Cette partie est organisée comme suit. Dans la deuxième section, nous étudions la variation de la solutionpar rapport aux données initiales, aux paramètres et aux processus directeurs. Dans la troisième section, nous montrons que, sous l'unicité forte, on peut construire des solutions fortes uniques par approximation polygonale plutôt que de faire appel au célèbre théorème de Yamada Watanabe. En utilisant des arguments similaires, nous montrons dans la dernière section que la propriété de l'unicité trajectorielle est une propriété générique au sens de Baire. Cela signifie que, au sens de Baire, la plupart des MVSDEs à coefficients continus bornés ont des solutions fortes uniques.

0.2.2 Hypothèses et préliminaires

Soit (B_t) un mouvement brownien d-dimensionnel défini sur un espace de probabilité $(\Omega, \mathcal{F}, P)$, équipé d'une filtration $(\mathcal{F}_t)$, satisfaisant les conditions habituelles. Tout au long de cet article, nous considérons l'équation différentielle stochastique de McKean-Vlasov (MVSDE) appelée aussi équation différentielle stochastique à champ moyen, de la forme

$$\begin{cases} dX_t = b(t, X_t, \mathbb{P}_{X_t})ds + \sigma(t, X_t, \mathbb{P}_{X_t})dB_t \\ X_0 = x \end{cases} \tag{8}$$

Supposons que les coefficients vérifient les conditions suivantes ;

$(\mathbf{H_1})$ Supposons que

$$b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d$$

$$\sigma : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$$

sont des fonctions mesurables et continues en (x, μ) uniformément en $t \in [0, T]$.

$(\mathbf{H_2})$ Il existe $C > 0$ tel que pour tout $t \in [0, T], x, \in \mathbb{R}^d$ et $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$|b(t, x, \mu)| \leq C\,(1 + |x|)\,,$$

$$|\sigma(t, x, \mu)| \leq C\,(1 + |x|)\,.$$

La définition suivante donne le concept d'unicité trajectorielle pour l'équation(8).

Definition 0.2 *Nous disons que l'unicité du chemin est valable pour l'équation (8) si X et X' sont deux solutions définies sur le même espace de probabilité $(\Omega, \mathcal{F}, P)$ avec un mouvement brownien commun (B), avec éventuellement des filtrations différentes telles que $P\,[X_0 = X_0'] = 1$, alors X et X' sont indistingables.*

0.2.3 Stabilité par rapport aux conditions initiales et aux coefficients

Dans cette section, nous démontrerons que, sous des hypothèses minimales sur les coefficients et l'unicité forte des solutions, la solution unique est continue par rapport à la condition initiale et aux coefficients..

Nous notons par (X_t^x) la solution unique de (4.1) correspondant à la condition initiale $X_0^x = x$.

$$\begin{cases} dX_t^x = b(t, X_t^x, \mathbb{P}_{X_t^x})dt + \sigma(t, X_t^x, \mathbb{P}_{X_t^x})dB_t \\ X_0^x = x. \end{cases}$$

Theorem 0.6 *Suppose que $b(t,x,\mu)$ et $\sigma(t,x,\mu)$ satisfaire* $(\mathbf{H_1})$, $(\mathbf{H_2})$.*Alors, si l'unicité forte est vérifiée pour l'équation ((8) alors l'application*
$\Phi : \mathbb{R}^d \longrightarrow L^2(\Omega, \mathcal{C}([0,T], \mathbb{R}^d))$
définie par $(\Phi(x)_t) = (X_t^x)$ *est continue.*

En utilisant les mêmes techniques, nous pouvons prouver la continuité de la solution de (8) par rapport à un paramètre. En particulier, la solution est continue par rapport aux coefficients. Considérons une famille de fonctions dépendant d'un paramètre λ, et considérons l'équation différentielle stochastique :

$$\begin{cases} dX_t^\lambda = \sigma\left(\lambda, t, X_t^\lambda, P_{X_t^\lambda}\right) dB_t + b\left(\lambda, t, X_t^\lambda, P_{X_t^\lambda}\right) dt \\ X_0^\lambda = \varphi(\lambda). \end{cases} \tag{9}$$

Theorem 0.7 *Supposons que $\sigma(\lambda,t,x)$ est $b(\lambda,t,x)$ sont continus. Supposons en outre que pour chaque $T>0$, et chaque ensemble compact K il existe $L>0$ stel que*

$i)\sup\limits_{t\leq T}\left(|\sigma(\lambda,t,x)| + |b(\lambda,t,x)|\right) \leq L\ (1+|x|)$ uniformément dans λ,
$ii)\lim\limits_{\lambda\to\lambda_0}\sup\limits_{x\in K}\sup\limits_{t\leq T}\left(|\sigma(\lambda,t,x,\mu) - \sigma(\lambda_0,t,x,\mu)| + |b(\lambda,t,x,\nu) - b(\lambda_0,t,x,\nu)|\right) = 0,$
iii) $\varphi(\lambda)$ *est continu à* $\lambda = \lambda_0$.

Si l'unicité forte est vérifiée pour l'équation **??** *au point* λ_0*, on a :*

$$\lim_{\lambda\to\lambda_0} E\left[\sup_{t\leq T}\left|X_t^\lambda - X_t^{\lambda_0}\right|^2\right] = 0\ ,\ \textit{pour chaque}\ T\geq 0\ .$$

0.2.4 Construction de solutions fortes

Soit $\Delta : 0 = t_0^n < t_1^n < \ldots.. < t_n^n = T$ une subdivision de l'intervalle $[0,T]$. Définissons l'approximation polygonale d'Euler pour l'équation (8) par :

$$X_\Delta(x,t) = x + \int_0^t b(\phi_\Delta(s), X_\Delta, P_{X_\Delta})ds + \int_0^t \sigma(\phi_\Delta(s), X_\Delta, P_{X_\Delta})dB_s$$

où $\phi_\Delta(s) = t_i$, si $t_i \leq s < t_{i+1}$ et $\|\Delta_n\| = \max\limits_i \left(t_{i+1}^n - t_i^n\right)$ et $X_\Delta = X_\Delta(x, \phi_\Delta(s))$

Theorem 0.8 *Supposons que* $(\mathbf{H_1})$ *et* $(\mathbf{H_2})$ *soient vérifiées, alors sous l'unicité trajectorielle de l'équation (8) nous avons :*

1) $\lim\limits_{\|\Delta_n\| \to 0} \sup\limits_{x \in K} E\left[\sup\limits_{t \leq T} |X_\Delta(x,t) - X(x,t)|^2\right] = 0$

2) Il existe une fonctionnelle mesurable $F : \mathbb{R}^d \times W_0^d \longrightarrow W^d$ *qui est adaptée telle que la solution unique* X_t *peut être écrit* $X(.) = F(X(0), B(.))$, *où* $W^d = C\left(\mathbb{R}_+, \mathbb{R}^d\right)$ *et* $W_0^d = \left\{w \in C\left(\mathbb{R}_+, \mathbb{R}^d\right) : w(0) = 0\right\}$ *sont équipés de leurs tribus Boreliennes et des filtrations des coordonnées.*

0.2.5 L'unicité trajectorielle est une propriété générique

Dans cette section, nous prouvons que l'ensemble des fonctions pour lesquelles l'unicité trajectorielle est vérifiée est un ensemble de deuxième catégorie au sens de Baire. Ce type de propriété s'appelle une propriété générique. Il indique qu'en gros, dans un sens topologique, la plupart des MVSDEs à coefficients continus et bornés ont la propriété d'unicité trajectorielle. Ce résultat étend les résultats connus pour les équations différentielles d'Itô[9, 10, **?**] aux EDS de McKean Vlasov.

Nous allons définir quelques notations.

$e(x, \sigma, b)$ représente l'équation (1) correspondant aux coefficients σ, b et et la donnée initiale x.

$$\mathbf{S}^2 = \left\{\xi : \mathbf{R}_+ \times \Omega \longrightarrow \mathbf{R}^d, \text{ continue, } E\left[\sup_{t \leq T} |\xi_t|^2\right] < +\infty\right\}$$

On définit une métrique sur $\mathbf{M}^2$ par :

$$d\left(\xi_1, \xi_2\right) = \left(E \sup_{0 \leq t \leq T} |\xi_t^1 - \xi_t^2|^2\right)^{\frac{1}{2}}$$

$(\mathbf{M}^2, d)$ est un espace métrique complet.

Soitr $\mathcal{C}_1$ l'ensemble des fonctions uniformément continues bornées $b : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}\left(\mathbb{R}^d\right) \longrightarrow \mathbb{R}^d$, muni de la métrique ρ_1 définie par :

$$\rho_1(b_1, b_2) = \sup_{(t,x,\mu)\in\mathbb{R}_+\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d)} |b_1(t,x,\mu) - b_2(t,x,\mu)|$$

Notons que la métrique ρ_1 est compatible avec la topologie de la convergence uniforme sur $\mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$.

Soit $\mathcal{C}_2$ soit l'ensemble des fonctions bornées continues $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ dotée de la métrique correspondante ρ_2.

Il est clair que $\mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$ est un espace métrique complet, alors $\mathfrak{R} = \mathcal{C}_1 \times \mathcal{C}_2$ doté de la métrique du produit est un espace métrique complet également.

Soit $\mathcal{L}$ le sous-ensemble de $\mathfrak{R}$ composé de fonctions $h(t,x,\mu)$ qui sont Lipschitziennes dans tous leurs arguments.

Le théorème suivant est très important dans cette section.

Theorem 0.9 *Toute fonction bornée uniformément continue dans un espace métrique est une limite uniforme de fonctions de Lipschitz globalement.*

Proposition 0.2 *$\mathcal{L}$ est un sous-ensemble dense dans $\mathfrak{R}$.*

La fonction d'oscillation

Pour $x \in \mathbb{R}^d$ et $(b,\sigma) \in \mathfrak{R}$, soit $\xi(x,b,\sigma)$ the solution of equation $e(x,b,\sigma)$.

Define the oscillation function as follows

$$D_1(x,b,\sigma) : \mathbb{R}^d \times \mathcal{R} \longrightarrow \mathbb{R}_+$$

$$D_1(x,b,\sigma) = \lim_{\delta\to 0} \sup\{d(\xi(x,b_1,\sigma_1), \xi(x,b_2,\sigma_2);\ \ (b_i,\sigma_i) \in \mathcal{L} \text{ and } \lambda((b,\sigma),(b_i,\sigma_i)) < \delta,\ i = 1,2\}$$

Proposition 0.3 *Si $x \in \mathbb{R}^d$ et (b,σ) sont des coefficients, $(b,\sigma) \in \mathcal{L}$, alors $D_1(x,b,\sigma) = 0$.*

Proposition 0.4 *La fonction d'oscillation D est semi-continue supérierement en tout point de l'ensemble $\mathbb{R}^d \times \mathcal{L}$.*

Proposition 0.5 *Soit (x,b,σ) dans $\mathbb{R}^d \times \mathcal{R}$ tel que $D_1(x,b,\sigma) = 0$, alors l'équation 8 au moins une solution forte.*

L'existence et l'unicité des solutions est une propriété générique

L résultat principal de cette section est donné par ce qui suit.

Theorem 0.10 *Le sous ensemble* $\mathcal{U}$ *xconstitué des coefficients* (σ, b) *pour lesquels l'équation (2.1) admet une solution forte unique contient un sous ensemble de seconde catégorie dans l'espace de Baire space* $\mathfrak{R}$*.*

Remark 0.3 *En utilisant des arguments similaires il n'est pas difficile de montrer que l'nesemble des coefficients* (b, σ) *pour lesquels le schema d'Euler de Picard convergent, est un sous esnemble résiduel dans l'espace de Baire des fonctions bornées uniformément continues* $(\mathfrak{R}, \lambda)$.

0.3 Convergence du schéma de Carathéodory

Dans cette partie nous étudions la convergence du schéma numérique de Carathéodory pour une classe de MVSDEs non linéaires. Ce schéma d'approximation est défini par une suite de solutions de MVSDEs avec de petits retards. Nous commençons, à prouver dans les conditions de Lipschitz que le schéma converge vers la solution unique de notre équation. La preuve est basée sur des estimations des solutions et les arguments habituels du calcul stochastique. Ce résultat étend ceux de [12] au cas des MVSDEs. Dans notre deuxième résultat principal, nous prouvons le résultat de la convergence, quelle que soit la condition sur les coefficients, garantissant l'unicité forte, à condition que les coefficients soient continus et satisfont une certaine condition de croissance linéaire. La preuve est basée sur la tension des processus considérés et le théorème de sélection de Skorokhod. Nous utilisons un résultat profond caractérisant la convergence en probabilité en termes de faible convergence de couples de sous-séquences. Contrairement au schéma d'approximations successives de Picard, qui est valable sous des hypothèses plutôt fortes sur les coefficients, le schéma de Carathéodory converge sous toutes les conditions garantissant l'unicité forte. Des résultats similaires ont été prouvés pour les équations différentielles stochastiques classiques d'Itô [31].

0.3.1 Preliminaires

Soit $(\Omega, \mathcal{F}, P)$un espace de probabilité complet, équipé d'une filtration $(\mathcal{F}_t)$, et (B_t) un $(\mathcal{F}_t, P)$ -Mouvement Brownien avec des valeurs dans $\mathbb{R}^d$. L'objet d'étude est l'équation différentielle stochastique de McKean-Vlasov (MVSDE) suivante.

$$\begin{cases} dX_t = b(t, X_t, \mathbb{P}_{X_t})ds + \sigma(t, X_t, \mathbb{P}_{X_t})dB_s \\ X_0 = x, \end{cases} \tag{10}$$

Nous considérons les hypothèses suivantes.

$(\mathbf{H_1})$

$$b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d$$
$$\sigma : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d,$$

sont mesurables continues en (x, μ) avec croissance linéaire. Il existe $C > 0$ tel que pour tout (t, x, μ) in $0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$:

$$|b(t, x, \mu)| + |\sigma(t, x, \mu)| \leq C\left(1 + |x| + W_2(\mu, \delta_0)\right).$$

$(\mathbf{H_2})$ b et σ sont Lipschitz en (x, μ) uniformément en t

Le schema numérique de Carathéodory

Notre objectif est l'étude de la convergence du schéma de Carathéodory défini par

$$\begin{cases} dX_t^n = b(t, X^n_{t-\frac{1}{n}}, \mathbb{P}_{X^n_{t-\frac{1}{n}}})dt + \sigma(t, X^n_{t-\frac{1}{n}}, \mathbb{P}_{X^n_{t-\frac{1}{n}}})dB_t, \quad if \quad t > 0 \\ X_t^n = x, \quad \text{if} \quad -\frac{1}{n} \leq t \leq 0. \end{cases} \tag{11}$$

ll est clair que pour chaque entier fixe n, X^n est bien défini comme une semi-martingale de carré intégrable continue, qui peut être construite récursivement sur les intervalles $[0, \frac{1}{n}), [\frac{1}{n}, \frac{2}{n}), [\frac{2}{n}, \frac{3}{n}), ...,$ Le point important sur le schéma d'itération, que nous considérons, est que chaque retard MVSDE peut être résolue explicitement par des intégrations successives sur des intervalles de longueur $1/n$.

0.3.2 La convergence du scheme de Carathéodory : Le cas Lipschitzien

Le résultat principal de cette section est donné par ce qui suit.

Theorem 0.11 *Soit* $\blacksquare$ *et* b *des fonctions satisfaisant aux conditions* $(\mathbf{H}_1)$ *et* $(\mathbf{H}_2)$ *et soit* (X^n) *la suite définie par le schéma Carathéodory 11. Alors, la suite* (X^n) *converge uniformément en probabilité vers la solution unique* X *de l'équation (10).*

Pour prouver le théorème, nous avons besoin du lemme technique suivant.

Lemma 0.2 *Soit* (X^n) *definie par (11) and* $0 < T < \infty$,*alors :*

1) *Pour tout* $0 \leq t \leq T, n \geq 1,\ E[\sup\limits_{-1/n \leq s \leq t} |X^n(s)|^2] \leq C_1 e^{C_2 t}.$.
2) *Pour tout* $-\frac{1}{n} \leq s, t \leq T$ *and* $n \geq 1,\ E[|X^n(t) - X^n(s)|^2] \leq C_3 |t - s|$.

   où C_1, C_2, C_3 *sont des constantes postives indépendantes de* n *et* $t \in [0, T]$.

Remark 0.4 *Le dernier théorème nous offre une alternative pour démontrer l'existence et l'unicité d'une solution forte. Notez que dans la preuve classique nous utilisons le théorème du point fixe ou le schéma d'approximation de Picard (voir [20, 45]).*

0.3.3 Convergence of the Carathéodory scheme : le cas continu

Au lieu de l'hypothèse de Lipschitz, nous supposons que (10) a la propriété d'unicité forte (voir [44] pour la définition précise).

Le résultat principal de cette section est le théorème suivant.

Theorem 0.12 (Main) *Supposons que* σ *et* b *satisfassent l'hypothèse* $(\mathbf{H_1})$ *.Alors si l'équation (10) admet une solution forte unique, la suite* (X^n) *définie par le schéma Carathéodory (11) converge en probabilité vers l'unique solution* X *de l'équation (10).*

La preuve du dernier théorème est basée sur le lemme simple mais profond suivant [37], qui caractérise la convergence en probabilité en termes de convergence faible de couples de sous-suites d'éléments aléatoires.

Lemma 0.3 *Soit* (E, d) *un espace polonais et* (X^n) *une suite de variables aléatoires à valeurs dans* E*. Les assertions suivantes sont alors équivalentes :*

1) (X^n) *converge en probabilité vers* X

2) Pour chaque couple de sous-suites $\left(X^l\right)$ *and* (X^m)*, il existe une sous-suite* $\nu_k := \left(X^{m(k)}, X^{l(k)}\right)$ *convergeant faiblement vers un élément aléatoire* ■ *supporté par la diagonale* $\{(x, y) \in E \times E : x = y\}$.

En utilisant les mêmes arguments, nous pouvons démontrer le corollaire suivant.

Corollary 0.1 *Supposons que* σ *et* b *satisfont à* $(\mathbf{H_1})$*. Alors, si l'équation (10) vérifie l'unicité en loi la suite* (X^n) *définie par le schéma Carathéodory (11) converge en loi vers l'unique solution* X *de l'équation (10).*

Dans ce qui suit, nous donnerons des exemples de conditions sous lesquelles notre résultat s'applique.

Corollary 0.2 *Supposons que les coefficients soient unidimensionnels,* σ *ne dépend pas de la mesure de probabilité* μ *et les coefficients de l'équation (10) satisfont à :*

*(**A1**) Il existe* $C > 0$*, tel que :* $|b(t, x, \mu) - b(t, x, \nu)| \leq CW_2(\mu, \nu)$

*(**A2**) Il existe* $K > 0$*, tel que pour tout* $x, y \in \mathbb{R}$ *,* $|\sigma(t, x) - \sigma(t, y)| \leq K|x - y|)$.

*(**A3**) Il existe une fonction strictement croissante* $\kappa(u)$ *sur* $[0, +\infty)$ *telle que* $\kappa(0) = 0$ *et* ■ *is concave satisfaisant* $\int\limits_{0^+} \kappa^{-1}(u)du = +\infty$*, de telle sorte que pour tout* $x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ *et* $\mu \in \mathcal{P}_1(\mathbb{R})$, $|b(t, x, \mu) - b(t, y, \mu)| \leq \kappa(|x - y|)$..

Alors la conclusion du Théorème 0.15. reste valide.

Proof. According to [8] the MVSDE (10) has a unique strong solution under $(\mathbf{A_1})$ and $(\mathbf{A}_2)$, then the result follows by Theorem 0.15.⊡

Corollary 0.3 *Supposons que l'équation (10) a la forme :*

$$X_t = x + \int_0^t b(s, X_s, E\left[\varphi_1(X_s)\right])ds + \int_0^t \sigma(s, X_s, E\left[\varphi_2(X_s)\right])dB_s$$

Supposons en outre que les coefficients satisfont aux conditions suivantes :

(B1) *b est mesurable, uniformément borné, Hölderien en la deuxième variable uniformément en la première et la troisième variables.* $b(t,x,.)$ *est différentiable avec une dérivée bornée et* φ_1 *est Hölderienne continu avec exposant* $0<\alpha\leq 1$.

(B2) $\sigma(t,.,.)$ *est globalement Lipschitz dans les deuxième et troisième variables uniformément en la première.*

$\sigma(t,x,.)$ *est différenciable dans la troisième variable et sa dérivée* $\sigma_y(t,x,.)$ *est bornée. De plus, la dérivée* $\sigma_y(t,x,.)$ *est Hölderienne continue par rapport à la deuxième variable* x.

(B3) Il existe$\lambda>1$ *tel que pour tout* $\xi\in\mathbb{R}^d$: $\frac{1}{\lambda}\leq|\langle\sigma\sigma^*\xi,\xi\rangle|\leq\lambda\,|\xi|^2$.

Alors la conclusion du Théorème 0.15 est valide.

Remark 0.5 *il est bien connu, même pour les équations différentielles ordinaires, que l'existence et l'unicité ne sont pas suffisantes pour la convergence du schéma d'approximations successives de Picard. Notre résultat offre donc une alternative au schéma de Picard. En effet, à la différence du schéma Picard, le schéma Carathéodory converge sous toutes les conditions assurant l'unicité forte.*

0.4 Compactification en contrôle optimal des MVSDEs

0.4.1 Introduction

Dans cette parte, nous considérons les problèmes de contrôle optimal, où le système est régi par une équation differentielle stochastique de McKean-Vlasov (MVSDE), également appelée équation différentielle stochastique à champ moyen (MFSDE) :

$$\begin{cases} dX_t = b\left(t, X_t, \mathbb{P}_{X_t}, u_t\right)dt + \sigma\left(t, X_t, \mathbb{P}_{X_t}, u_t\right)dB_t \\ X_0 = x \end{cases} \tag{12}$$

La fonction de coût sur l'intervalle de temps $[0,T]$ a la forme :

$$J(u) = E\left[\int_0^T h(t, X_t, \mathbb{P}_{X_t}, u_t)dt + g(X_T, \mathbb{P}_{X_T})\right]. \tag{13}$$

L'objectif est de minimiser le coût fonctionnel $J(u)$ sur l'espace $\mathcal{U}_{ad}$, trouver u^* tel que $J(u^*) = \min\{J(u), u \in \mathcal{U}_{ad}\}$.

Nous prouvons deux résultats principaux. Le premier est un résultat d'approximation fort du problème de contrôle relaxé par une suite de problèmes de contrôle stricts. Cela signifie en particulier que la relaxation du problème initial n'affecte pas la fonction de valeur de notre problème. Cela signifie que les problèmes de contrôle stricts et relaxés ont en fait la même fonction de valeur. Ceci est effectué sous des coefficients simplement continus et l'unicité trajectorielle des solutions de l'équation d'état. Ce résultat pourrait être considéré comme un résultat de stabilité par rapport à la variable de contrôle et pourrait être utilisé pour effectuer des approximations numériques. Le deuxième résultat principal est l'existence d'un contrôle relaxé optimal sous continuité des coefficients. Les principaux ingrédients utilisés dans les preuves sont les critères de la tension et le théorème de sélection de Skorokhod.

0.4.2 Formulation du problème

Soit (B_t) est un $d-$dimensionnel mouvement brownien, défini sur un espace de probabilité $(\Omega, \mathcal{F}, P)$, équipé d'une filtration $(\mathcal{F}_t)$, satisfaisant aux conditions habituelles. Soit $\mathbb{A}$ un espace métrique compact appelé espace d'actions.

Supposons les conditions suivantes :

$(\mathbf{H_1})$

$$\begin{aligned}
&b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{A} \longrightarrow \mathbb{R}^d \\
&\sigma : [0,T] \times \mathbb{R}^d \times \mathcal{P}\left({}_2\mathbb{R}^d\right) \times \mathbb{A} \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d \\
&h : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{A} \longrightarrow \mathbb{R}^d \\
&g : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d
\end{aligned} \tag{14}$$

sont des fonctions bornées et continues.

Considérons le problème de contrôle suivant

$$\begin{cases}
\text{Minimiser } J(u) = E\left[\int_0^T h(t, X_t, \mathbb{P}_{X_t}, u_t)dt + g(X_T, \mathbb{P}_{X_T})\right] \\
X_t = x + \int_0^t b\left(s, X_s, \mathbb{P}_{X_s}, u_s\right) ds + \int_0^t \sigma\left(s, X_s, \mathbb{P}_{X_s}, u_s\right) dB_s
\end{cases}$$

sur l'ensemble $\mathcal{U}_{ad}$ des contrôles admissibles (appelés contrôles stricts) qui sont des processus progressivement mesurables avec des valeurs sur l'espace des actions $\mathbb{A}$.

Soit L^{ν} le générateur infinitésimal associé à la solution de l'équation (12),

$$L^{\nu} f(t,x,a) = \frac{1}{2}\sigma\sigma^{*}\frac{\partial^2 f}{\partial x^2}(t,x,a) + b\frac{\partial f}{\partial x}(t,x,a),$$

où $b = b(t,x,\nu,a)$ et $\sigma\sigma^{*} = \sigma\sigma^{*}(t,x,\nu,a)$ pour $\nu \in \mathcal{P}_2(\mathbb{R})$.

Definition 0.3 *Un contrôle strict est le terme* $\alpha = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P, u, X, x)$ *tel que*

(1) $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ *est un espace de probabilité équipé d'une filtration* $(\mathcal{F}_t)_{t\geq 0}$ *satisfaisant aux conditions habituelles.*

(2) u *est un* $\mathbb{A}$*-processus valorisé, progressivement mesurable par rapport à* $(\mathcal{F}_t)$*.*

(3) (X_t) *est* $\mathbb{R}^d$*-valorisé,* $\mathcal{F}_t$*-adapté, avec des trajets continus, tels pour chaque*

$$f(X_t) - f(x) - \textstyle\int_0^t L^{\mathbb{P}_{X_s}} f\,(s, X_s, u_s)\,ds \quad \text{est un } P - \text{martingale}, \tag{15}$$

for every $f \in C_b^2$*.*

0.4.3 Le problème du contrôle relaxé

Martingale measures

Definition 0.4 *Soit* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ *être un espace de probabilité filtré et* $(E, \mathcal{E})$ *un espace métrique* $\{M_t(A), t \geqq 0, A \in \mathcal{E}\}$ *est appelée un* $\mathcal{F}_t-$*mesure de martingale si et seulement si :*

1) $\{M_t(A), t \geqq 0\}$ *est un* $\mathcal{F}_t$*-martingale,* $\forall A \in \mathcal{E}$*;*

2) $\forall t > 0, M_t(.)$ *est mesure* $\sigma-$*finie dans le sens suivant : il existe une suite non décroissante* (E_n) *de* E *avec* $\cup_n E_n = E$ *tel que :*

a) Pour chaque $t > 0$, $\sup\limits_{A\in\mathcal{E}_n} E\left[M(A,t)^2\right] < \infty$, $\mathcal{E}_n = \mathcal{B}(E_n)$

b) Pour chaque $t > 0$, $E\left[M(A_j,t)^2\right] \to 0$, *pour toute suite* (A_j) *de* $\mathcal{E}_n$ *décroissant à* $\emptyset$*.*

Pour $A, B \in \mathcal{E}$, il existe un unique process prévisible $\langle M(A), M(B)\rangle_t$, tel que

$$M(A,t)M(B,t) - \langle M(A), M(B)\rangle_t \text{ est une martingale.}$$

Une mesure de martingale M est appelé orthogonale si $M(A,t).M(B,t)$ est une martingale pour $A, B \in \mathcal{E}$, $A \cap B = \emptyset$.
Si M est une mesure de martingale orthogonale, on peut prouver l'existence d'une mesure positive $\sigma-$finie $\mu(ds, dx)$ on $\mathbb{R} \times E$, $\mathcal{F}_t-$prévisible, de telle sorte que pour chaque B de $\mathcal{A}$ le processus $(\mu((0,t] \times B))_t$ est prévisible et satisfait

$$\forall B \in \mathcal{E}, \forall t > 0, \quad \mu((0,t] \times B) = \langle M(B)\rangle_t \quad P - a.s.$$

On se réfère à [69] et [29] pour une construction complète de l'intégrale stochastique par rapport aux mesures de martingales orthogonales.

Le problème du contrôle relaxé

Soit $\mathbb{V}$ être l'ensemble des mesures du produit μ sur $[0,T] \times \mathbb{A}$ dont la projection sur $[0,T]$ coïncide avec la mesure de Lebesgue dt. $\mathbb{V}$ comme un sous-espace fermé de l'espace des mesures positives du radon $\mathbb{M}_+([0,T] \times \mathbb{A})$ est compact pour la topologie de convergence faible.

Definition 0.5 *Un contrôle à valeurs mesures sur l'espace de probabilité filtré* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ *est une variable aléatoire* $\mu = dt.\mu_t(da)$ *à valeurs dans* $\mathbb{V}$, *telle que* $\mu_t(da)$ *est progressivement mesurable par rapport à* $(\mathcal{F}_t)$ *et tel que pour chaque* t, $1_{(0,t]}.\mu$ *est* $\mathcal{F}_t-$*mesurable.*

Definition 0.6 *Un contrôle relaxé est le terme* $\alpha = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P, \mu, X, x)$ *tel que*
(1) $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ *est un espace de probabilité équipé d'une filtration* $(\mathcal{F}_t)_{t\geq 0}$ *satisfaisant aux conditions habituelles ;*
(2) μ *est progressivement mesurable par rapport à* $(\mathcal{F}_t)$ *;*

(3) (X_t) *est* $\mathbb{R}^d$*-valorisé,* $\mathcal{F}_t$*-adapté, continu, tel que*

$$f(X_t) - f(X_0) - \int_0^t \int_{\mathbb{A}} L^{\mathbb{P}_{X_s}} f(s, X_s, a)\, \mu_s(da) ds \text{ est une } P - \text{martingale}, \tag{16}$$

pour chaque $f \in C_b^2$.

La fonction de coût relaxé est définie par

$$J(u) = E\left[\int_0^T \int_{\mathbb{A}} h(t, X_t, \mathbb{P}_{X_t}, a)\mu_t(da)dt + g(X_T, \mathbb{P}_{X_T})\right]$$

Le processus d'état correspondant à un contrôle relaxé doit satisfaire une MVSDE dirigée par uen mesure martingale.

$$\begin{cases} dX_t = \int_{\mathbb{A}} b(t, X_t, \mathbb{P}_{X_t}, a)\, \mu_t(da)dt + \int_{\mathbb{A}} \sigma(t, X_t, \mathbb{P}_{X_t}, a)\, M(da, dt) \\ X_0 = x, \end{cases} \tag{17}$$

où M est une mesure de martingale continue orthogonale, avec intensité $dt\mu_t(da)$.

Le problème de contrôle relaxé est donc défini comme suit.

$$\begin{cases} \text{Mimimiser } J(\mu) = E\left[\int_0^T \int_{\mathbb{A}} h(t, X_t, \mathbb{P}_{X_t}, u_t)\mu_t(da)dt + g(X_T, \mathbb{P}_{X_T})\right] \\ \text{sujet à } X_t = x + \int_0^t \int_{\mathbb{A}} b\left(s, X_s, \mathbb{P}_{X_s}, a\right) \mu_s(da)ds + \int_0^t \int_{\mathbb{A}} \sigma\left(s, X_s, \mathbb{P}_{X_s}, u_s\right) M(da, ds) \end{cases}$$

sur la classe des contrôles relaxés.

0.4.4 Approximation du problème de contrôle relaxé

Lemma 0.4 (chattering lemma) *Soit* μ *être un contrôle relaxé. Il existe alors une suite de processus adaptés* (u^n) *à valeur dans* $\mathbb{A}$*, de telle sorte que la suite de mesures aléatoires* $\left(\delta_{u_t^n}(da)\, dt\right)$ *converge dans* $\mathbb{V}$ *vers* $dt.\mu_t(da)$*,* $P - a.s.$*,c'est à dire pour tout* f *continue sur* $[0, T] \times \mathbb{A}$*, on a :*

$$\lim_{n \to +\infty} \int_0^T f(s, u_s^n)ds = \int_0^T \int_{\mathbb{A}} f(s, a)\mu_t(da) \text{ , } P - a.s.$$

Soit μ *un contrôle relaxé et* X_t le processus d'état relaxé qui est la solution du MVSDE correspondant au contrôle relaxé μ :

$$\begin{cases} dX_t = \int_{\mathbb{A}} \sigma\left(t, X_t, \mathbb{P}_{X_t}, a\right) M(da, dt) + \int_{\mathbb{A}} b\left(t, X_t, \mathbb{P}_{X_t}, a\right) \mu_t(da) dt \\ X_0 = x. \end{cases} \tag{18}$$

Soit (u^n) être la suite de contrôles stricts approximant μ comme dans le dernier lemme et X_t^n être la solution de l'équation d'état (12) correspond à u^n. Si nous désignons $M^n(t, F) = \int_0^t \int_F \delta_{u_s^n}(da) dW_s$, alors $M^n(t, F)$ est une mesure de martingale orthogonale et X_t^n peut être écrit sous une forme relaxée comme suit

$$\begin{cases} dX_t^n = \int_{\mathbb{A}} b(t, X_t^n, \mathbb{P}_{X_t^n}, a) \delta_{u_t^n}(da) dt + \int_{\mathbb{A}} \sigma(t, X_t, \mathbb{P}_{X_t^n}, a) M^n(dt, da) \\ X_0 = x \end{cases}$$

$$J(\mu^n) = E\left[\int_0^T \int_{\mathbb{A}} h(t, X_t^n, \mathbb{P}_{X_t^n}, u_t) \mu_t^n(da) dt + g(X_T^n, \mathbb{P}_{X_T^n})\right]$$

Le résultat principal de cette section est donné par.

Theorem 0.13 *Soit X être la solution de processus d'état relaxé de (17) et X^n la solution de (4.1) correspond à u^n. Alors, si l'unicité trajectorielle est vérifiée pour l'équation (17) on a :*

1) $\lim\limits_{n \to +\infty} E\left[\sup\limits_{t \leq 1} |X_t^n - X_t|^2\right] = 0.$

2) Si $J(u^n)$ et $J(\mu)$ désignent les coûts attendus correspondant respectivement à u^n et μ, alors $\left(J\left(u^n\right)\right)$ converge vers $J\left(\mu\right)$.

0.4.5 Existence de contrôles optimaux

Existence de contrôles relaxés optimaux

En utilisant les mêmes techniques on peut montrer l'existence d'un contrôle relaxé optimal.

Theorem 0.14 *On suppose $(\boldsymbol{H_1})$, alors le problème de contrôle relaxé admet une solution optimale.*

Existence de contrôles optimaux stricts

On définit la classe des contrôles relaxés ayant la forme

$$\nu_t = \sum_{i=1}^{p} \alpha_i(t) dt \delta_{u_i(t)}(da), u_i(t) \in \mathbb{A}, \alpha_i(t) \geq 0 \text{ et } \sum_{i=1}^{p} \alpha_i(t) = 1. \tag{19}$$

où $\alpha_i(t)$ et $u_i(t)$ sont des processus stochastiques adaptés.

Notez que si ν_t a la forme (4.14) alors le processus d'état contrôlé relaxé est la solution de

$$\begin{cases} dX_t &= \sum_{i=1}^{p} \alpha_i(t) b(t, X_t, \mathbb{P}_{X_t}, u_i(t)) dt + \sum_{i=1}^{p} \alpha_i(t)^{1/2} \sigma(t, X_t, \mathbb{P}_{X_t}, u_i(t)) dW_t^i \\ X_0 &= x \end{cases} \tag{20}$$

Proposition 0.6 *Soit μ être un contrôle relaxé et X^μ la solution de processus d'état correspondante de (17). Il existe alors un contrôle de la forme*

$$\nu_t = \sum_{i=1}^{p} \alpha_i(t) \delta_{u_i(t)}(da), \;\; u_i(t) \in A, \;\; \alpha_i(t) \geq 0 \;\; et \; \sum_{i=1}^{p} \alpha_i(t) = 1 \tag{21}$$

tel que :

1) $X^\mu = X^\nu$.

2) $J(\mu) = J(\nu)$.

Corollary 0.4 *Supposons que l'ensemble suivant*

$$P(t, X_t) = \left\{ \widetilde{b}(t, X_t, \mathbb{P}_{X_t}, a); \;\; a \in \mathbb{A} \right\} \subset \mathbb{R}^N \tag{22}$$

est convexe, où $\widetilde{b}(t, X_t, \mathbb{P}_{X_t}, a) = \left(b(t, X_t, \mathbb{P}_{X_t}, a), \;\; \sigma\sigma^*(t, X_t, \mathbb{P}_{X_t}, a), \;\; h(t, X_t, \mathbb{P}_{X_t}, a)\right)$, alors le problème de contrôle strict admet une solution optimale.

Remark 0.6 *En utilisant les mêmes techniques on peut démontrer le même résultat pour des problèmes de contrôle singulier de MVSDEs.*

Introduction

The general framework of this thesis is stochastic analysis and optimal control of systems governed by stochastic differential equations. These are dynamical systems evolving randomly in time, which could be seen as generalization of ordinary differential equations.The main flavor of stochastic calculus is the Itô calculus, which has been introduced by K. Itô, in the 40s to build the paths associated with Markov processes, associated with elliptic operators. To do so, he invented the stochastic calculus that bears his name, to give meaning to stochastic integrals and stochastic differential equations. Stochastic calculus has found many applications such as in PDEs theory, engineering, biology and economics. More recently, the most spectacular application is its role in mathematical finance. Robert Merton introduced stochastic calculus in the study of finance and at the same time Fischer Black and Myron Scholes developed their option pricing formula. Nowadays mathematical finance became one of the fastest growing interdisciplinary sciences. It draws from the disciplines of probability theory, statistics, scientific computing and partial differential equations, to provide models and derive relationships between asset prices, market movements and interest rates.

In this Thesis our attention will be focused on a special class of stochastic differential equations, called Mc Kean-Vlasov stochastic differential equations (MVSDE) or mean-field stochastic differential equations (MFSDE).

These are SDEs given as follows

$$\begin{cases} dX_t = b(t, X_t, P_{X_t})dt + \sigma(t, X_t, P_{X_t})dW_t \\ X_0 = x. \end{cases}$$

where b is the drift and σ is the diffusion coefficient and (W_t) is a Brownian motion. Note that in MFSDEs the coefficients depend not only on the state variable but also on its marginal distribution . This brings an additional difficulty compared to classical Itô SDEs. The solutions of such equation are known in the literature as non linear diffusions.

MFSDEs were first investigated in statistical physics by M. Kac [46] as a stochastic counterpart for the Vlasov equation of plasma [68]. The study of such equation has been performed by Mc Kean, see Snitzman [66], see for an excellent introduction to this area. These equations were obtained as limits of some weakly interacting particle systems as the number of particles tends to infinity. This kind of approximation result is called "propagation of chaos", which says that when the number of particles tends to infinity, the equations defining the evolution of the particles could be replaced by a single equation, called the McKean-Vlasov equation. This mean-field equation, represents in some sense the average behavior of the infinite number of particles, see [45, 66] for details.

Recently there has been a renewed interest for Mc Kean Vlasov SDEs, in the context of mean-field games (MFG) theory. This theory was invented by P.L. Lions and J.M. Lasry in 2006 [51], to solve the problem of existence of an approximate Nash equilibrium for differential games, with a large number of players. Since the earlier papers [51, 43], mean-field game theory has raised a lot of interest, motivated by applications to various fields such as game theory, mathematical finance, communications networks and management of oil ressources.

Control of Mc Kean Vlasov SDEs is a relatively new field in stochastic control theory. It is basically motivated by the behaviour of large populations in interaction and has applications in economics and finance in particular the study of systemic risk. Other applications in social networks, physics (statistical physics) and biology. In particular the well known Markowitz mean-variance problem portfolio selection problem is a typical example of Mc Kean Vlasov control problem where one should minimize an objective function involving a quadratic function of the expectation, due to the variance term. The main drawback, when dealing with such mean-field stochastic control problems, is that the Bellmann principle of optimality does not hold. For this kind of problems, the stochastic maximum principle, provides a powerful tool to solve them. One can refer also to the most updated reference on the subject [20] and the complete list of references therein.

This thesis contains four papers, dealing with various questions of existence, uniqueness and stability of MVSDEs as well as optimal control of such equations. In the sequel, we describe in some details the contents of these papers.

In the first paper, we study some properties of MVSDEs such as existence, uniqueness, and stability properties. In particular, we prove an existence and uniqueness theorem for a class of MVSDEs under Osgood type condition on the coefficients, improving the well known globally Lipschitz case. Moreover, we investigate the stability with respect to initial conditions, coefficients and driving processes, which are continuous martingales and bounded variation processes. These properties will be investigated under Lipschitz condition with respect to the state variable and the distribution and generalize known properties for classical Itô SDEs. Furthermore, we prove that in the context of stochastic control of systems driven by MVSDEs, the relaxed and strict control problems have the same value function. The idea is to embedd the space of strict controls into the set of measure valued controls, called relaxed controls, which enjoys good compactness properties. Under the Lipschitz condition we prove that the value functions are equal. Note that this result extends to general Mc Kean Vlasov equations known results [6, 7] established for a special class of MVSDEs, where the dependence of the coefficient on the distribution variable is made via a linear form of the distribution.

In the second paper, our attention willl be focused on strong stability properties of the solution of the MVSDE under pathwise uniqueness of solutions and merely continuous coefficients. Since the coefficients are only continuous without additional regularity, one cannot expect to apply Gronwall's lemma. Instead of Gronwall's lemma, we use tightness arguments and the famous Skorokhod selection theorem to prove the desired convergence results. In particular we show that the Euler polygonal scheme is convergent provided that there is pathwise uniqueness. This provides us with an effective way to construct strong solutions for MVSDEs. Moreover, we prove that the solution is stable under small perturbation of the initial condition and coefficients. Our results generalize similar ones proved for classical Itô SDEs [9, 37, 47].

Furthermore, we show that the set of bounded uniformly continuous coefficients for which strong existence and uniqueness hold is a generic property in the sense of Baire. This means that in the sense of Baire category, most of MVSDEs with bounded uniformly continuous coefficients have unique solutions. This last result extends in particular [2, 9, 10] to MVSDEs. The third paper is devoted to the convergence of the Carathéodory numerical scheme for a class of nonlinear MVSDEs. This approximate scheme is defined by a sequence of solutions of MVSDEs with small delays. We start by proving under Lipschitz conditions, that the scheme converges to the unique solution of the MVSDE. The proof is based on estimates of the solutions and the usual arguments from stochastic calculus. This result extends those in ([**?**]) to MVSDEs. In our second main result, we prove the convergence result, under any condition on the coefficients, ensuring pathwise uniqueness, provided the coefficients are continuous and satisfy some linear growth condition. The proof is based on the tightness of the processes under consideration and the Skorokhod embedding theorem. We use a deep result charactezing the convergence in probability in terms of weak convergence of couples of subsequences. Unlike the Picard successive approximation scheme, which is valid under rather strong assumptions on the coefficients, the Carathéodory scheme converges under any conditions ensuring the pathwise uniqueness.

The fourth paper is concerned by the study of existence, as well as some approximation properties of optimal controls. Our starting point is a control problem (called strict control problem), which does not necessarily admit an optimal control. We construct a second control problem, called the relaxed control problem, with two main properties. The first one is that the relaxed problem admits an optimal solution. The second property is that the relaxed controls as well as their states and cost functionals could be approximated by means of strict controls and their states and cost functionals. This last stability property is useful in numerical and engineering problems. Indeed, it is more convenient to handle nearly optimal controls which are functions instead of optimal controls which are measure valued processes. To achieve this objective we apply the so-called compactification method to show the existence of an optimal relaxed control. This method is based on the relative compactness of the distributions of the processes under consideration, which does not require any regularity of the coefficients or the value function.

We prove two main results. The first is a strong approximation result of the relaxed control problem by a sequence of strict control problems. This means in particular that relaxing out the initial problem does not affect the value function of our problem.This is performed under merely continuous coefficients and pathwise uniqueness of the solutions of the state equation. This result could be seen as a stability result with respect to the control variable and could be used to perform numerical approximations. The second main result is the existence of an optimal relaxed control under continuity of the coefficients. Note that our results improve known existence of optimal controls for systems driven by Itô SDEs [5, 9] as well as McKean-Vlasov SDEs [6, 7]. The main ingredients used in the proofs are the tightness criteria and Skorokhod selection theorem.

Chapitre 1

Stability of McKean–Vlasov stochastic differential equations and applications

(Joint work with K. Bahlali and B. Mezerdi)

ABSTRACT. We consider McKean-Vlasov stochastic differential equations (MVSDEs), which are SDEs where the drift and diffusion coefficients depend not only on the state of the unknown process but also on its probability distribution. This type of SDEs was studied in statistical physics and represents the natural setting for stochastic mean-field games. We will first discuss questions of existence and uniqueness of solutions under an Osgood type condition improving the well known Lipschitz case. Then we derive various stability properties with respect to initial data, coefficients and driving processes, generalizing known results for classical SDEs. Finally, we establish a result on the approximation of the solution of a MVSDE associated to a relaxed control by the solutions of the same equation associated to strict controls. As a consequence, we show that the relaxed and strict control problems have the same value function. This last property improves known results proved for a special class of MVSDEs, where the dependence on the distribution was made via a linear functional.

Key words : McKean-Vlasov stochastic differential equation – Stability – Martingale measure - Wasserstein metric – Existence – Mean-field control – Relaxed control.

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1.1 Introduction

We will investigate some properties of a particular class of stochastic differential equations (SDE), called Mc Kean-Vlasov stochastic differential equations (MVSDE) or mean-field stochastic differential equations. These are SDEs described by

$$\begin{cases} dX_t = b(t, X_t, \mathbb{P}_{X_t})ds + \sigma(t, X_t, \mathbb{P}_{X_t})dB_s \\ X_0 = x, \end{cases}$$

where b is the drift, σ is the diffusion coefficient and (B_t) is a Brownian motion. For this type of equations the drift and diffusion coefficient depend not only on the state variable X_t, but also on its marginal distribution $\mathbb{P}_{X_t}$. This fact brings a non trivial additional difficulty compared to classical Itô SDEs. The solutions of such equation are known in the literature as non linear diffusions.

MVSDEs were first studied in statistical physics by M. Kac [46], as a stochastic counterpart for the Vlasov equation of plasma [68]. The probabilistic study of such equation has been performed by H.P. Mc Kean [56], see [66] for an introduction to this research field. These equations were obtained as limits of some weakly interacting particle systems as the number of particles tends to infinity. This convergence property is called in the literature as the propagation of chaos. The MVSDE, represents in some sense the average behavior of the infinite number of particles. One can refer to [20, 35, 45] for details on the existence and uniqueness of solutions for such SDEs, see also [17, 18] for the case of Mc Kean Vlasov backward stochastic differential equations (MVBSDE). Existence and uniqueness with less regularity on the coefficients have been established in [21, 23, 24, 38, 59, 64]. Recently there has been a renewed interest for MVSDEs, in the context of mean-field games (MFG) theory, introduced independently by P.L. Lions and J.M. Lasry [51] and Huang, Malhamé Caines [43] in 2006. MFG theory has been introduced to solve the problem of existence of an approximate Nash equilibrium for differential games, with a large number of players (see [?]). Since the earlier papers, MFG theory and mean-field control theory has raised a lot of interest, motivated by applications to various fields such as game theory, mathematical finance, communications networks and management of oil ressources. One can refer to the most recent and updated reference on the subject [20] and the complete bibliographical list therein.

Our main objective in this paper is to study some properties of such equations such as existence, uniqueness, and stability properties. In particular, we prove an existence and uniqueness theorem for a class of MVSDEs under Osgood type condition on the coefficients, improving the well known globally Lipschitz case. It is well known that stability properties of deterministic or stochastic dynamical systems are crucial in the study of such systems. It means that the trajectories do not change too much under small perturbations. We study stability with respect to initial conditions, coefficients and driving processes, which are continuous martingales and bounded variation processes. These properties will be investigated under Lipschitz condition with respect to the state variable and the distribution and generalize known properties for classical Itô SDEs, see [9, 44]. Furthermore, we prove that in the context of stochastic control of systems driven by MVSDEs, the relaxed and strict control problems have the same value function. As it is well known when the Filipov type convexity condition is not fulfilled, there is no mean to prove the existence of a strict control. The idea is then to embedd the usual strict controls into the set of measure valued controls, called relaxed controls, which enjoys good compactness properties. So for the relaxed control to be a true extension of the initial problem, the value functions of both control problems must be the same. Under the Lipschitz condition we prove that the value functions are equal. Note that this result extends to general Mc Kean Vlasov equations known results [6, 7] established for a special class of MVSDEs, where the dependence of the coefficient on the distribution variable is made via a linear form of the distribution.

1.2 Formulation of the problem and preliminary results

1.2.1 Assumptions

Let $(\Omega, \mathcal{F}, P)$ be a probability space, equipped with a filtration $(\mathcal{F}_t)$, satisfying the usual conditions and (B_t) a d-dimensional $(\mathcal{F}_t, P)$ $-$Brownian motion. Let us consider the following

Mc Kean-Vlasov stochastic differential equation called also mean-field stochastic differential equation (MVSDE)

$$\begin{cases} dX_t = b(t, X_t, \mathbb{P}_{X_t})ds + \sigma(t, X_t, \mathbb{P}_{X_t})dB_s \\ X_0 = x \end{cases} \tag{1.1}$$

Note that for this kind of SDEs, the drift b and diffusion coefficient σ depend not only on the position, but also on the marginal distribution of the solution.

The following assumption will be considered throughout this paper.

Let us denote $\mathcal{P}_2(\mathbb{R}^d)$ the space of probability measures with finite second order moment. That is for each $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ $\int |x|^2 \mu(dx) < +\infty$.

$\mathbf{(H_1)}$ Assume that

$$b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d$$
$$\sigma : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$$

are Borel measurable functions and there exist $C > 0$ such that for every $(t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$:

$$|b(t, x, \mu)| + |\sigma(t, x, \mu)| \leq C\,(1 + |x|)$$

$\mathbf{(H_2)}$ There exist $L > 0$ such that for any $t \in [0, T], x,\ x' \in \mathbb{R}^d$ and $\mu,\ \mu' \in \mathcal{P}_2(\mathbb{R}^d)$,

$$|b(t, x, \mu) - b(t, x', \mu')| \leq L[|x - x'| + W_2(\mu, \mu')]$$
$$|\sigma(t, x, \mu) - \sigma(t, x', \mu')| \leq L[|x - x'| + W_2(\mu, \mu')]$$

where W_2 denotes the 2-Wasserstein metric.

1.2.2 Wasserstein metric

Let $\mathcal{P}(\mathbb{R}^d)$ be the space of probability measures on $\mathbb{R}^d$ and for any $p > 1$, denote by $\mathcal{P}_p(\mathbb{R}^d)$ the subspace of $\mathcal{P}(\mathbb{R}^d)$ of the probability measures with finite moment of order p.

For $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$, define the p-Wasserstein distance $W_p(\mu, \nu)$ by :

$$W_p(\mu,\nu) = \inf_{\pi \in \Pi(\mu,\nu)} [\int_{E\times E} |x-y|^p \, d\pi(x,y)]^{1/p}$$

where $\Pi(\mu, \nu)$ denotes the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ whose first and second marginals are respectively μ and ν.

In the case $\mu = \mathbb{P}_X$ and $\nu = \mathbb{P}_Y$ are the laws of $\mathbb{R}^d$-valued random variable X and Y of order p, then

$$W_p(\mu,\nu)^p \leq \mathbb{E}[\,|X-Y|^p].$$

Indeed

$$\begin{aligned} W_p(\mu,\nu) &= \inf_{\pi \in \Pi(\mu,\nu)} [\int_{E\times E} |x-y|^p \, d\pi(x,y)]^{1/p} \\ W_p(\mu,\nu)^p &= \inf_{\pi \in \Pi(\mu,\nu)} [\int_{E\times E} |x-y|^p \, d\pi(x,y)] \\ &\leq \int_{E\times E} |x-y|^p \, d(\mathbb{P}_{(X,Y)}(x,y) \\ &= \mathbb{E}[\,|X-Y|^p] \end{aligned}$$

In the literature the Wasserstein metric is restricted to W_2 while W_1 is often called the Kantorovich-Rubinstein distance because of the role it plays in optimal transport.

1.3 Existence and uniqueness of solutions

1.3.1 The globally Lipschitz case

The following theorem states that under global Lipschitz condition, (1.1) admits a unique solution. Its complete proof is given in [66] for a drift depending linearly on the law of X_t that is $b(t,x,\mu) = \int_{\mathbb{R}^d} b'(t,x,y)\mu(dy)$ and a constant diffusion. The general case as in (1.1) is treated in [20] Theorem 4.21 or [45] Proposition 1.2 and is based on a fixed point theorem on the space of continuous functions with values in $\mathcal{P}_2(\mathbb{R}^d)$. Note that in [35, 45] the authors consider MVSDEs driven by general Lévy process instead of a Brownian motion.

Theorem 1.1 *Under assumptions* $(\mathbf{H_1})$, $(\mathbf{H_2})$, *(1.1) admits a unique solution such that* $E[\sup_{t\leq T}|X_t|^2] < +\infty$

Proof

Let us give the outline of the proof. Let $\mu \in \mathcal{P}_p(\mathbb{R}^d)$ be fixed, the classical Itô's theorem gives the existence and uniqueness of a solution denote by (X_t^μ) satisfying $E[\sup_{t\leq T}|X_t^\mu|^2] < +\infty$. Now let us consider the mapping

$$\Psi : \mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}^d)) \longrightarrow \mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}^d))$$
$$\mu \longrightarrow \Psi(\mu) = (\mathcal{L}(X_t^\mu))_{t\geq 0}, \text{ the distribution of } X_t^\mu.$$

Ψ is well defined as X_t^μ has continuous paths and $E[\sup_{t\leq T}|X_t^\mu|^2] < +\infty$.

To prove the existence and uniqueness of(1.1), it is sufficient to prove that the mapping Ψ has a unique fixed point. By using usual arguments from stochastic calculus and relation and the property of Wasserstein metric it is easy to show that :

$$\sup_{t\leq T} W_2(\left(\Psi^k(\mu)\right)_t, \left(\Psi^k(\nu)\right)_t)^2 \leq C\frac{T^k}{k!}\sup_{t\leq T} W_2(\mu_t,\nu_t)^2$$

For large k, Ψ^k is a strict contraction which implies that Ψ admits a unique fixed point in the complete metric space $\mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}^d))$.$\boxdot$

The following version MVSDEs is also considered in the control literature

$$\begin{cases} dX_t = b(t, X_t, \int \varphi(y)\mathbb{P}_{X_t}(dy))dt + \sigma(t, X_t, \int \psi(y)\mathbb{P}_{X_t}(dy))dW_t \\ X_0 = x \end{cases} \tag{1.2}$$

where

$(\mathbf{H_3})$ b, σ, φ and ψ are Borel measurable bounded functions such that $b(t,.,.)$, $\sigma(t,.,.)$, φ and ψ are globally lipshitz functions in $\mathbb{R}^d \times \mathbb{R}^d$.

Proposition 1.1 *Under assumptions* $(\mathbf{H_1})$ *and* $(\mathbf{H_3})$*the MVSDE (1.2) has a unique strong solution. Moreover for each* $p > 0$ *we have* $E(|X_t|^p) < +\infty$.

Proof

Let us define $\overline{b}(t,x,\mu)$ and $\overline{\sigma}(t,x,\mu)$ on $[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$ by

$$\overline{b}(t,x,\mu)=b(.,.,\int\varphi(x)d\mu(x),.),\ \overline{\sigma}(t,x,\mu)=\sigma(t,x,\int\psi(x)d\mu(x)).$$

According to the last Theorem it is sufficient to check that $\overline{b}$ and $\overline{\sigma}$ are Lipschitz in (x,μ). Indeed since the coefficients b and σ are Lipschitz continuous in x, then $\overline{b}$ and $\overline{\sigma}$ are also Lipschitz in x. Moreover one can verify easily that $\overline{b}$ and $\overline{\sigma}$ are also Lipshitz continuous in μ, with respect to the Wasserstein metric

$$\begin{aligned}&W_2(\mu,\nu)=\inf\left\{\left(E^Q\left|X-Y\right|^2\right)^{1/2};Q\in\mathcal{P}_2(\mathbb{R}^d\times\mathbb{R}^d),\ \text{with marginals }\mu,\nu\right\}\\&=\sup\left\{\int hd\left(\mu-\nu\right);\ \ |h(x)-h(y)|\leq|x-y|\right\},\end{aligned}$$

Note that the second equality is given by the Kantorovich-Rubinstein theorem [20]. Since the mappings b and φ in the the MFSDE are Lipschitz continuous in y we have

$$\begin{aligned}&\left|b(.,.,\int\varphi(y)d\mu(y),.)-b(.,.,\int\varphi(y)d\nu(y),.)\right|\\&\leq K\left|\int\varphi(y)d(\mu(y)-\nu(y))\right|\\&\leq K'.W_2\left(\mu,\nu\right)\end{aligned}$$

Therefore $\overline{b}(t,.,.)$ is Lipschitz continuous in the variable $(x,\mu)\in\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$ uniformly in $t\in[0,[T]]$

Similar arguments can be used for σ. $\boxdot$

1.3.2 The uniqueness under Osgood type condition

In this section we relax the global Lipschitz condition in the state variable. We will prove the existence and uniqueness of a solution when the coefficients are globally Lipschitz in the distribution variable and satisfy an Osgood type condition in the state variable. To be more precise let us consider the following MVSDE

$$\begin{cases}dX_t=b(t,X_t,\mathbb{P}_{X_t})ds+\sigma(t,X_t)dB_s\\ X_0=x\end{cases}\tag{1.3}$$

Assume that b and σ are real valued bounded Borel measurable functions satisfying :

$(\mathbf{H_4})$ There exist $C > 0$, such that for every $x \in \mathbb{R}$ and $(\mu, \nu) \in \mathcal{P}_1(\mathbb{R}) \times \mathcal{P}_1(\mathbb{R})$:

$$|b(t, x, \mu) - b(t, x, \nu)| \leq CW_1(\mu, \nu)$$

$(\mathbf{H_5})$There exists a strictly increasing function $\rho(u)$ on $[0, +\infty)$ such that $\rho(0) = 0$ and ρ^2 is convex satisfying $\int_{0^+} \rho^{-2}(u)du = +\infty$, such that for every $(x, y) \in \mathbb{R} \times \mathbb{R}$ and $\mu \in \mathcal{P}_2(\mathbb{R})$, $|\sigma(t, x) - \sigma(t, y)| \leq \rho(|x - y|)$.

$(\mathbf{H_6})$ There exists a strictly increasing function $\kappa(u)$ on $[0, +\infty)$ such that $\kappa(0) = 0$ and κ is concave satisfying $\int_{0^+} \kappa^{-1}(u)du = +\infty$, such that for every $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R})$, $|b(t, x, \mu) - b(t, y, \mu)| \leq \kappa(|x - y|)$.

In the next Theorem we derive the pathwise uniqueness for (1.3) under an Osgood type condition in the state variable. This result improves [44],Theorem 3.2, established for classical Itô's SDEs and [20] Theorem 4.21, at least for MVSDEs with a diffusion coefficient not depending on the distribution variable.

Theorem 1.2 *Under assumptions* $(\mathbf{H_4}) - (\mathbf{H_6})$, *the MVSDE (1.3) enjoys the property of pathwise uniqueness.*

Proof

The following proof is inspired from [20] Theorem 4.21.

Since $\int_{0^+} \rho^{-2}(u)du = +\infty$, there exist a decreasing sequence (a_n) of positive real numbers such that $1 > a_1$

satisfying

$$\int_{a_1}^{1} \rho^{-2}(u)du = 1, \int_{a_2}^{a_1} \rho^{-2}(u)du = 2, ..., \int_{a_n}^{a_{n-1}} \rho^{-2}(u)du = n, ...$$

Clearly (a_n) converges to 0 as n tends to $+\infty$.

The properties of ρ allow us to construct a sequence of functions $\psi_n(u)$, $n = 1, 2, \ldots$, such that

i) $\psi_n(u)$ is a continuous function such that its support is contained in (a_n, a_{n-1})

ii) $0 \leq \psi_n(u) \leq \frac{2}{n}\rho^{-2}(u)$ and $\int\limits_{a_n}^{a_{n-1}} \psi_n(u)du = 1$

Let $\varphi_n(x) = \int\limits_0^{|x|} dy \int\limits_0^{y} \psi_n(u)du,\ x \in \mathbb{R}$

It is clear that $\varphi_n \in \mathcal{C}^2(\mathbb{R})$ such that $|\varphi_n'| \leq 1$ and (φ_n) is an increasing sequence converging to $|x|$.

Let X_t^1 and X_t^2 two solutions of corresponding to the same Brownian motion and the same MVSDE

$$X_t^1 - X_t^2 = \int\limits_0^t (\sigma(s, X_s^1) - \sigma(s, X_s^2))dW_s + \int\limits_0^t \left(b(s, X_s^1, \mathbb{P}_{X_s^1}) - b(s, X_s^2, \mathbb{P}_{X_s^2})\right)dW_s$$

By using Itô's formula we obtain

$$\begin{aligned}
\varphi_n(X_t^1 - X_t^2) = \; & \int\limits_0^t \varphi_n'(X_s^1 - X_s^2)\left(\sigma(s, X_s^1) - \sigma(s, X_s^2)\right)dW_s \\
& + \int\limits_0^t \varphi_n'(X_s^1 - X_s^2)\left(b(s, X_s^1, \mathbb{P}_{X_s^1}) - b(s, X_s^2, \mathbb{P}_{X_s^2})\right) ds \\
& + \frac{1}{2}\int\limits_0^t \varphi_n''(X_s^1 - X_s^2)\left(\sigma(s, X_s^1) - \sigma(s, X_s^2)\right)^2 ds
\end{aligned}$$

φ_n' and σ being bounded, then the process under the sign integral is sufficiently integrable. Then the first term is a true martingale, so that its expectation is 0. Therefore

$$\begin{aligned}
E\left(\varphi_n(X_t^1 - X_t^2)\right) = \; & E\left[\int\limits_0^t \varphi_n'(X_s^1 - X_s^2)\left(b(s, X_s^1, \mathbb{P}_{X_s^1}) - b(s, X_s^2, \mathbb{P}_{X_s^2})\right) ds\right] \\
& + \frac{1}{2}E\left[\int\limits_0^t \varphi_n''(X_s^1 - X_s^2)\left(\sigma(s, X_s^1) - \sigma(s, X_s^2)\right)^2 ds\right] \\
& = I_1 + I_2
\end{aligned}$$

But we know that $W_1(\mathbb{P}_{X_s^1}, \mathbb{P}_{X_s^2}) = E\left(|X_s^1 - X_s^2|\right)$

Then

$$|I_1| \leq E\int\limits_0^t \kappa(|X_s^1 - X_s^2|)ds + \int\limits_0^t CE\left(|X_s^1 - X_s^2|\right) ds$$

Then by Growall lemma, there exist a constant M such that $|I_1| \leq M.E\int_0^t \kappa(|X_s^1 - X_s^2|)ds$

On the other hand

$$\begin{aligned} |I_2| = &\ \frac{1}{2}E\left[\int_0^t \varphi_n''(X_s^1 - X_s^2)\left(\sigma(s,X_s^1) - \sigma(s,X_s^2)\right)^2 ds\right] \\ &\leq \frac{1}{2}E\left[\int_0^t \frac{2}{n}\rho^{-2}(X_s^1 - X_s^2)\rho^2(X_s^1 - X_s^2)ds\right] = \frac{t}{n} \end{aligned}$$

Then $|I_2|$ tends to 0 as n tends to $+\infty$.

Letting n tending to $+\infty$ it holds that : $E\left(|X_t^1 - X_t^2|\right) \leq M.E\int_0^t \kappa(|X_s^1 - X_s^2|)ds$. Since $\int_{0^+} \kappa^{-1}(u)du = +\infty$ we conclude that $E\left(|X_t^1 - X_t^2|\right) = 0$.⊡

Remark. *The continuity and boundness of the coefficients imply the existence of a weak solution (see [45] Proposition 1.10). Then by the well known Yamada - Watanabe theorem applied to equation (1.3) (see [48] example 2.14, page 10), the pathwise uniqueness proved in the last theorem implies the existence and uniqueness of a strong solution.*

1.4 Convergence of the Picard successive approximation

Assume that $b(t,x,\mu)$ and $\sigma(t,x,\mu)$ satisfy assumptions $(\mathbf{H_1})$, $(\mathbf{H_2})$. We will prove the convergence of the Picard iteration scheme. This scheme is useful for numerical computations of the unique solution of (1.1). Let $(X_t^0) = x$ for all $t \in [0,T]$ and define $\left(X_t^{n+1}\right)$ as the solution of the following SDE

$$\begin{cases} dX_t^{n+1} = b(t, X_t^n, \mathbb{P}_{X_t^n})dt + \sigma(t, X_t^n, \mathbb{P}_{X_t^n})dB_t \\ X_0^{n+1} = x \end{cases}$$

Theorem 1.3 *Under assumptions* $(\mathbf{H_1})$, $(\mathbf{H_2})$, *the sequence* (X^n) *converges to the unique solution of (1.1)*

$$E[\sup_{t\leq T}|X_t^n - X_t|^2] \to 0$$

Proof. Let $n \geq 0$, by applying usual arguments such as Schwartz inequality and Burkholder-Davis Gundy inequality for the martingale part, we get

$$
\begin{aligned}
|X_t^{n+1} - X_t^n|^2 &\leq 2(\int_0^t |b(s, X_s^n, P_{X_s^n}) - b(s, X_s^{n-1}, P_{X_s^{n-1}})|ds)^2 \\
&\quad + 2(\int_0^t |\sigma(s, X_s^n, P_{X_s^n}) - \sigma(s, X_s^{n-1}, P_{X_s^{n-1}})|dB_s)^2 \\
E[\sup_{t\leq T} |X_t^{n+1} - X_s^n|^2] &\leq 2TE[\int_0^T |b(s, X_s^n, \mathbb{P}_{X_s^n}) - b(s, X_s^{n-1}, \mathbb{P}_{X_s^{n-1}})|^2 ds] \\
&\quad + 2C_2 E[\int_0^T |\sigma(s, X_s^n, \mathbb{P}_{X_s^n}) - \sigma(s, X_s^{n-1}, \mathbb{P}_{X_s^{n-1}})|^2 ds]
\end{aligned}
$$

the coefficients b and σ being Lipschitz continuous in (x, μ) we get

$$
\begin{aligned}
E[\sup_{t\leq T} |X_t^{n+1} - X_t^n|^2] &\leq 2(T + C_2)L^2 \int_0^T E[|X_s^n - X_s^{n-1}|^2] + W_2(\mathbb{P}_{X_s^n}, \mathbb{P}_{X_s^{n-1}})ds \\
&\leq 4(T + C_2)L^2 \int_0^T E[|X_s^n - X_s^{n-1}|^2]ds \\
&\leq 4(T + C_2)L^2 \int_0^T E[\sup_{t\leq T} |X_s^n - X_s^{n-1}|^2]ds
\end{aligned}
$$

Then for all $n \geq 1$, and $t \leq T$

$$
\begin{aligned}
E[\sup_{t\leq T} |X_t^1 - X_s^0|^2] &\leq 2T \int_0^T b|(s, x, \mu)|^2 ds + C_2 \int_0^T \sigma|(s, x, \mu)|^2 ds \\
&\leq 2(C_2 + T)M(1 + E(|x|^2))T \\
&\leq A_1 T
\end{aligned}
$$

where the constant A_1 only depends on C_2, M, T and $E[|x|^2]$. So by induction on n we obtain

$$
E[\sup_{t\leq T} |X_t^{n+1} - X_t^n|^2] \leq \frac{A_2^{n+1} T^{n+1}}{(n+1)!}
$$

This implies in particular that (X_t^n) is a Cauchy sequence in $L^2(\Omega, \mathcal{C}([0,T], \mathbb{R}^d))$ which is complete. Therefore (X_t^n) converges to a limit (X_t) which is the unique solution of (1.1) $\boxdot$

1.5 Stability with respect to initial condition

In this section, we will study the stability of MFSDEs with respect to small perturbations of the initial condition.

We denote by (X_t^x) the unique solution of (1.1) such that $X_0^x = x$

$$\begin{cases} dX_t^x = b(t, X_t^x, \mathbb{P}_{X_t^x})dt + \sigma(t, X_t^x, \mathbb{P}_{X_t^x})dB_t \\ X_0^x = x \end{cases}$$

Theorem 1.4 *Assume that $b(t,x,\mu)$ and $\sigma(t,x,\mu)$ satisfy $(\mathbf{H_1})$, $(\mathbf{H_2})$, then the mapping*

$$\Phi : \mathbb{R}^d \longrightarrow L^2(\Omega, \mathcal{C}([0,T], \mathbb{R}^d))$$

defined by $(\Phi(x)_t) = (X_t^x)$ is continuous.

Proof

Let (x_n) be a sequence in $\mathbb{R}^d$ converging to x. Let us prove that $\lim_{n \longrightarrow +\infty} E\left[\sup_{t \leq T} |X_t^n - X_t|^2\right] = 0$, where $X_t^n = X_t^{x_n}$. We have

$$\begin{aligned} |X_t^n - X_t|^2 = |x_n - x &+ \int_0^t (b(s, X_s^n, \mathbb{P}_{X_s^n}) - b(s, X_s^n, \mathbb{P}_{X_s^n}))ds \\ &+ \int_0^t (\sigma(s, X_s^n, \mathbb{P}_{X_s^n}) - \sigma(s, X_s^n, \mathbb{P}_{X_s^n}))dB_s|^2 \end{aligned}$$

$$\begin{aligned} &\leq 3|x_n - x|^2 + 3(\int_0^t |b(s, X_s^n, \mathbb{P}_{X_s^n}) - b(s, X_s^n, \mathbb{P}_{X_s^n})|ds)^2 \\ &+ 3(\int_0^t |\sigma(s, X_s^n, \mathbb{P}_{X_s^n}) - \sigma(s, X_s^n, \mathbb{P}_{X_s^n})|dB_s)^2 \end{aligned}$$

$$E\left[\sup_{t\leq T}|X_t^n - X_t|^2\right] \leq 3|x_n - x|^2 + 3E[\sup_{s\leq t}\int_0^t |b(s, X_s^n, \mathbb{P}_{X_s^n}) - b(s, X_s^n, \mathbb{P}_{X_s^n})|ds]^2$$
$$+ 3E[\sup_{s\leq t}\int_0^t |\sigma(s, X_s^n, \mathbb{P}_{X_s^n}) - \sigma(s, X_s^n, \mathbb{P}_{X_s^n})|dB_s]^2$$

we apply Schwartz and Burkholder Davis Gundy inequalities to obtain

$$E\left[\sup_{t\leq T}|X_t^n - X_t|^2\right] \leq 3|x_n - x|^2 + 3TE[\int_0^t |b(s, X_s^n, \mathbb{P}_{X_s^n}) - b(s, X_s^n, \mathbb{P}_{X_s^n})|^2 ds]$$
$$+ 3C_2E[\int_0^t |\sigma(s, X_s^n, \mathbb{P}_{X_s^n}) - \sigma(s, X_s^n, \mathbb{P}_{X_s^n})|^2 ds]$$

The Lipschitz condition implies that

$$E\left[\sup_{t\leq T}|X_t^n - X_t|^2\right] \leq 3|x_n - x|^2 + 3(T + C_2)L^2[\int_0^t E|X_s^n - X_s|^2 + W_2(\mathbb{P}_{X_s^n}, \mathbb{P}_{X_s})]ds$$

Since

$$W_2^2(\mathbb{P}_{X_t^n}, \mathbb{P}_{X_t}) \leq E[|X_s^n - X_s|^2],$$

then

$$E\left[\sup_{t\leq T}|X_t^n - X_t|^2\right] \leq 3|x_n - x|^2 + 6(T + c_2)L^2\int_0^t E|X_s^n - X_s|^2 ds$$
$$\leq 3|x_n - x|^2 + 6(T + c_2)L^2\int_0^t E[\sup_{t\leq T}|X_s^n - X_s|^2]ds.$$

Finally we apply Gronwall lemma to conclude that

$$E\left[\sup_{t\leq T}|X_t^n - X_t|^2\right] \leq 3|x_n - x|^2 \exp[6(T + c_2)L^2 t]$$

Therefore $\lim_{n\to\infty} x_n = x$ implies that $\lim_{n\to\infty} E\left[\sup_{t\leq T}|X_t^n - X_t|^2\right] = 0.\boxdot$

1.6 Stability with respect to the coefficients

In this section, we will establish the stability of the MVSDE with respect to small perturbation of the cofficients b and σ. Let us consider sequences of functions (b_n) and (σ_n) and consider the corresponding MFSDE :

$$\begin{aligned} dX_t^n &= b_n(t, X_t^n, \mathbb{P}_{X_t^n})dt + \sigma_n(t, X_t^n, \mathbb{P}_{X_t^n})dB_t \\ X_0^n &= x \end{aligned} \tag{1.4}$$

The following theorem gives us the continuous dependence of the solution with respect to the coefficients.

Theorem 1.5 *Assume that the functions $b(t, x, \mu)$, $b_n(t, x, \mu)$, $\sigma(t, x, \mu)$ and $\sigma_n(t, x, \mu)$ satisfy $(\mathbf{H_1})$, $(\mathbf{H_2})$. Further suppose that for each $T > 0$, and each compact set K there existe $C > 0$ such that*

i) $\sup_{t\leq T}(|b_n(t, x, \mu)| + |\sigma_n(t, x, \mu)|) \leq C(1 + |x|)$,

ii) $\lim_{n\to\infty} \sup_{t\leq T} \sup_{x\in K} \sup_{\mu\in\mathcal{P}_2(\mathbb{R}^d)} ||b_n(t, x, \mu) - b(t, x, \mu)|| + ||\sigma_n(t, x, \mu) - \sigma(t, x, \mu)|| = 0$

then

$$\lim_{n\to\infty} E\left[\sup_{t\leq T} |X_t^n - X_t|^2\right] = 0$$

where (X_t^n) and (X_t) are respectively solutions of (1.4) and (1.1).

Proof

For each $n \in \mathbb{N}$, let (X_t^n) be a solution of (1.4), then by using

$$
\begin{aligned}
|X_t^n - X_t|^2 \leq 3(&\int_0^t |b_n(s, X_s^n, \mathbb{P}_{X_s^n}) - b_n(s, X_s, \mathbb{P}_{X_s})|ds)^2 \\
&+ 3(\int_0^t |b_n(s, X_s, \mathbb{P}_{X_s}) - b(s, X_s, \mathbb{P}_{X_s})|ds)^2 \\
&+ 3\left|\int_0^t \left(\sigma_n(s, X_s^n, \mathbb{P}_{X_s^n}) - \sigma_n(s, X_s, \mathbb{P}_{X_s})\right) dB_s\right|^2 \\
&+ 3\left|\int_0^t \left(\sigma_n(s, X_s, \mathbb{P}_{X_s}) + \sigma(s, X_s, \mathbb{P}_{X_s})\right) dB_s\right|^2
\end{aligned}
$$

By using the Lipschitz continuity and Burkholder Davis Gundy inequality, it holds that

$$
\begin{aligned}
E\left[\sup_{t\leq T} |X_t^n - X_t|^2\right] &\leq 3(T+C_2)L^2 \int_0^t E[|X_s^n - X_s|^2] + W_2(\mathbb{P}_{X_s^n}, \mathbb{P}_{X_s})^2]ds \\
&+ 3(T+C_2)E[\int_0^t |b_n(s, X_s, \mathbb{P}_{X_s}) - b(s, X_s, \mathbb{P}_{X_s})|^2 ds] \\
&+ 3(T+C_2)E[\int_0^t |\sigma_n(s, X_s, \mathbb{P}_{X_s}) - \sigma(s, X_s, \mathbb{P}_{X_s})|^2 ds] \\
&\leq 6(T+C_2)L^2 \int_0^T E[|X_s^n - X_s|^2]ds + K_n \\
&\leq 6(T+C_2)L^2 \int_0^T E\left[\sup_{s\leq t} |X_s^n - X_s|^2\right] dt + K_n
\end{aligned}
$$

such that

$$
K_n = 3(T+C_2)E[\int_0^T \left(|b_n(s, X_s, \mathbb{P}_{X_s}) - b(s, X_s, \mathbb{P}_{X_s})|^2 + |\sigma_n(s, X_s, \mathbb{P}_{X_s}) + \sigma(s, X_s, \mathbb{P}_{X_s})|^2\right) ds]
$$

An application of Gronwall lemma allows us to get

$$
E\left[\sup_{t\leq T} |X_t^n - X_t|^2\right] \leq K_n \exp 6(T+C_2)L^2.T
$$

By using assumptions i) and ii) it is easy to see that $K_n \longrightarrow 0$.as $n \longrightarrow +\infty$,which achieves the proof.$\boxdot$

1.7 Stability with respect to the driving processes

In this section, we consider McKean-Vlasov SDE driven by continuous semi-martingales.

Let $b : [0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ and $\sigma : [0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d\times d}$ be bounded continuous functions.

We consider MVSDEs driven by continuous semi-martingales of the following form

$$\begin{cases} dX_t = b(t, X_t, \mathbb{P}_{X_t})dA_t + \sigma(t, X_t, \mathbb{P}_{X_t})dM_t \\ X_0 = x \end{cases} \tag{1.5}$$

where A_t is an adapted continuous process of bounded variation and M_t is a continuous local martingale.

Let us consider the following sequence of MVSDEs

$$\begin{cases} dX_t^n = b(t, X_t^n, \mathbb{P}_{X_t^n})dA_t^n + \sigma(t, X_t^n, \mathbb{P}_{X_t^n})dM_t^n \\ X_0^n = x \end{cases} \tag{1.6}$$

where (A^n) is a sequence of $\mathcal{F}_t$-adapted continuous process of bounded variaton and M^n is continuous $(\mathcal{F}_t, \mathbb{P})$-local martingales.

Let us assume that (A, A^n, M, M^n) satisfy :

$(\mathbf{H_7})$

1) The family (A, A^n, M, M^n) is bounded in $\mathbb{C}([0,1])^4$.

2) $(M^n - M)$ converges to 0 in probability in $\mathbb{C}([0,1])$ as n tends to $+\infty$.

3)The total variation $(A^n - A)$ converges to 0 in probability as n tends to $+\infty$.

Theorem 1.6 *Let $b(t,x,\mu)$ and $\sigma(t,x,\mu)$ satisfy $(\mathbf{H_1})$, $(\mathbf{H_2})$. Further assume that (A, A^n, M, M^n) satisfy $(\mathbf{H_7})$. Then for each $\varepsilon > 0$*

$$\lim_{n\to\infty} E[\sup_{t\le T} |X_t^n - X_t|^2] = 0$$

where (X_t^n) and (X_t) are respectively solutions of (1.6) and (1.5).

Proof

Let $n \in \mathbb{N}$, then by using similar arguments as in the preceding theorems, we have

$$\begin{aligned}\mathbb{E}[\sup_{t\leq T}|X_t^n - X_t|^2] &\leq 3(E[\sup_{t\leq T}\int_0^t |b(s,X_s^n,\mathbb{P}_{X_s^n}) - b(s,X_s^n,\mathbb{P}_{X_s^n})|]dA_s^n)^2 \\ &+ 3(E\left[\sup_{t\leq T}\left|\int_0^t \left(\sigma(s,X_s^n,\mathbb{P}_{X_s^n}) - \sigma(s,X_s^n,\mathbb{P}_{X_s^n})\right) dM_s^n\right|^2\right] \\ &+ 3E[(\sup_{t\leq T}\int_0^t |b(t,X_t,\mathbb{P}_{X_t})|\, d\,|A_s^n - A_s|)^2 + \sup_{t\leq T}\left|\int_0^t \sigma(t,X_t,\mathbb{P}_{X_s})d(M_s^n - M_s)\right|^2]\end{aligned}$$

Let

$$K_n = 3E[(\sup_{t\leq T}\int_0^t |b(t,X_t,\mathbb{P}_{X_t})|\, d\,|A_s^n - A_s|)^2 + \sup_{t\leq T}\left|\int_0^t \sigma(t,X_t,\mathbb{P}_{X_s})d(M_s^n - M_s)\right|^2]$$

By using Schwartz and Burkholder Davis Gundy inequalities along with the Lipschitz condition, we obtain

$$\begin{aligned}\mathbb{E}[\sup_{t\leq T}|X_t^n - X_t|^2] &\leq C(T)[\int_0^T (E\left(\sup_{s\leq t}|X_s^n - X_s|^2\right) + \mathbb{W}_2(\mathbb{P}_{X_s^n},\mathbb{P}_{X_s})^2)(dA_s^n + d < M^n, M^n >_s)] + K_n \\ &\leq 2C(T)\int_0^T E[\sup_{s\leq t}|X_s^n - X_s|^2](dA_s^n + d < M^n, M^n >_s) + K_n\end{aligned}$$

where $C(T)$ is a positive constant which may change from line to line.

Since $(A_s^n + d < M^n, M^n >_s)$ is an increasing process, then according to the Stochastic Gronwall lemma [58] Lemma 29.1, page 202, we have

$$\mathbb{E}[\sup_{t\leq T}|X_t^n - X_t|^2] \leq 2K_nCE(A_T^n + < M^n, M^n >_T)) < +\infty,$$

where C is a constant.

By using assumption $(\mathbf{H_7})$ it is easy to that

$$\lim_{n\to\infty} K_n = 0$$

Therefore

$$\lim_{n\to\infty} \mathbb{E}[\sup_{t\leq T} |X_t^n - X_t|^2] = 0$$

$\boxdot$

1.8 Approximation of relaxed control problems

It is well known that in the deterministic as well as in stochastic control problems , an optimal control does not necessarily exist in the space of strict controls, in the absence of convexity conditions. The classical method is then to introduce measure valued controls which describe the introduction of a stochastic parameter see [30] and the references therein. These measure valued controls called relaxed controls generalize the strict controls in the sense that the set of strict controls may be identified as a dense subset of the set of the relaxed controls. The relaxed control problem is a true extension of the strict control problem if they have the same value function. That is the infimum among strict controls is equal to the infimum among relaxed controls. This last property is based on the continuity of the dynamics and the cost functional with respect to the control variable. We show that under Lipschitz condition and continuity with respect to the control variable of the coefficients that the strict and relaxed control problems have the same value function. Our result extends those in [6, 7], to general MFSDEs of the type 1.7.

Let $\mathbb{A}$ be some compact metric space called the action space. A strict control (u_t) is a measurable, $\mathcal{F}_t-$ adapted process with values in the action space $\mathbb{A}$. We denote $\mathcal{U}_{ad}$ the space of strict controls.

The state process corresponding to a strict control is the unique solution, of the following MFSDE

$$\begin{cases} dX_t = b(t, X_t, \mathbb{P}_{X_t}, u_t)ds + \sigma(t, X_t, \mathbb{P}_{X_t}, u_t)dB_s \\ X_0 = x \end{cases} \tag{1.7}$$

and the corresponding cost functional is given by

$$J(u) = E\left[\int_0^T h(t, X_t, \mathbb{P}_{X_t}, u_t dt + g(X_T, \mathbb{P}_{X_T})\right].$$

The problem is to minimize $J(u)$ over the space $\mathcal{U}_{ad}$ of strict controls and to find $u^* \in \mathcal{U}_{ad}$ such that $J(u^*) = \inf\{J(u), u \in \mathcal{U}_{ad}\}$.

Let us consider the following assumptions in this section.

$(\mathbf{H_4})$ $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{A} \longrightarrow \mathbb{R}^d$, $\sigma : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{A} \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$, are continuous bounded functions .

$(\mathbf{H_5})$ $b(t,.,.,a)$ and $\sigma(t,.,.,a)$ are Lipschitz continuyous uniformly in $(t,a) \in [0,T] \times \mathbb{A}$.

$(\mathbf{H_6})$ $h : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{A} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, are bounded continuous functions, such that $h(t,.,.,a)$ is Lipschitz in (x,μ).

It is clear that under assumptions $(\mathbf{H_4})$ and $(\mathbf{H_5})$ and according to Theorem 3.1 , for each $u \in \mathcal{U}_{ad}$, the MFSDE (1.7) has a unique strong solution, such that for every $p > 0$, $E(|X_t|^p) < +\infty$. Moreover for each $u \in \mathcal{U}_{ad}$ $|J(u)| < +\infty$.

Let $\mathbb{V}$ be the set of product measures μ on $[0,T] \times \mathbb{A}$ whose projection on $[0,T]$ coincides with the Lebesgue measure dt. $\mathbb{V}$ as a closed subspace of the space of positive Radon measures $\mathbb{M}_+([0,T] \times \mathbb{A})$ is compact for the topology of weak convergence.

Definition 1.1 *A relaxed control on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a random variable $\mu = dt.\mu_t(da)$ with values in $\mathbb{V}$, such that $\mu_t(da)$ is progressively measurable with respect to $(\mathcal{F}_t)$ and such that for each t, $1_{(0,t]}.\mu$ is $\mathcal{F}_t-$measurable.*

Remark 1.1 *The set $\mathcal{U}_{ad}$ of strict controls is embedded into the set of relaxed controls by identifying u_t with $dt\delta_{u_t}(da)$.*

It was proved in [29] for classical control problems and in [7] that the relaxed state process corresponding to a relaxed control must satisfy a MFSDE driven by a martingale measure instead of a Brownian motion. That is the relaxed state process satisfies

$$\begin{cases} dX_t = \int_{\mathbb{A}} b(t, X_t, \mathbb{P}_{X_t}, a)\, \mu_t(da)dt + \int_{\mathbb{A}} \sigma(t, X_t, \mathbb{P}_{X_t}, a)\, M(da, dt) \\ X_0 = x, \end{cases} \tag{1.8}$$

where M is an orthogonal continuous martingale measure, with intensity $dt\mu_t(da)$.Using the same tools as in Theorem 3.1, it is not difficult to prove that (1.8) admits a unique strong solution. The following Lemma, known in the control literature as Chattering Lemma states that the set of strict controls is a dense subset in the set of relaxed controls.

Lemma 1.1 *i) Let* (μ_t) *be a relaxed control. Then there exists a sequence of adapted processes* (u_t^n) *with values in* $\mathbb{A}$*, such that the sequence of random measures* $\left(\delta_{u_t^n}(da)\, dt\right)$ *converges in* $\mathbb{V}$ *to* $\mu_t(da)\, dt$*,* $P-a.s.$

ii) For any g *continuous in* $[0,T] \times \mathbb{M}_1(\mathbb{A})$ *such that* $g(t,.)$ *is linear, we have*
$\lim\limits_{n\to+\infty} \int_0^t g(s, \delta_{u_s^n})ds = \int_0^t g(s, \mu_s)ds$ *uniformly in* $t \in [0,T]$ *,* $P-a.s.$

Proof. See [30] $\boxdot$

Let X_t^n be the solution of the state equation*(*1.7) corresponding to u^n, where u^n is a strict control defined as in the last Lemma. If we denote $M^n(t,F) = \int_0^t \int_F \delta_{u_s^n}(da)dW_s$, then $M^n(t,F)$ is an orthogonal martingale measure and X_t^n may be written in a relaxed form as follows

$$\begin{cases} dX_t^n = \int\limits_{\mathbb{A}} b(t, X_t^n, \mathbb{P}_{X_t^n}, a)\delta_{u_t^n}(da)dt + \int\limits_{\mathbb{A}} \sigma(t, X_t, \mathbb{P}_{X_t^n}, a)M^n(dt, da) \\ X_0 = x \end{cases}$$

Therefore X_t^n may be viewed as the solution of *(*1.8) corresponding to the relaxed control $\mu^n = dt\delta_{u_t^n}(da)$.

Since $\left(\delta_{u_t^n}(da)\, dt\right)$ *converges weakly to* $\mu_t(da)\, dt$, $P-a.s.$, then for every bounded predictable process $\varphi : \Omega \times [0,T] \times \mathbb{A} \to \mathbb{R}$, such that $\varphi(\omega, t, .)$ is continuous, we have

$$E\left[\left(\int_0^T \int_{\mathbb{A}} \varphi(\omega,t,a)M^n(dt,da) - \int_0^t \int_{\mathbb{A}} \varphi(\omega,t,a)M(dt,da)\right)^2\right] \to 0 \text{ } as \text{ } n \longrightarrow +\infty. \tag{1.9}$$

see ([5, 57]).

The following proposition gives the continuity of the dynamics (1.8) with respect to the control variable.

Proposition 1.2 *i) If X_t, X_t^n denote the solutions of state equation (1.8) corresponding to μ and μ^n, then For eacht $\leq T$, $\lim\limits_{n\rightarrow+\infty} E(|X_t^n - X_t|^2) = 0$.*
ii) Let $J(u^n)$ and $J(\mu)$ be the expected costs corresponding respectively to u^n and μ, then $\left(J\left(u^n\right)\right)$ converges to $J\left(\mu\right)$.

Proof. 1) Let X_t, X_t^n the solutions of the MVSDE (1.8) corresponding to μ and u^n. We have

$$
\begin{aligned}
|X_t - X_t^n| \;\; & \leq \left| \int_0^t \int_{\mathbb{A}} b\left(s, X_s, \mathbb{P}_{X_t}, u\right) \mu_s(da).ds - \int_0^t \int_{\mathbb{A}} b\left(s, X_s^n, \mathbb{P}_{X_t^n}, u\right) \delta_{u_s^n}(da) ds \right| \\
& + \left| \int_0^t \int_{\mathbb{A}} \sigma\left(s, X_s, \mathbb{P}_{X_t}, a\right) M(ds, da) - \int_0^t \int_{\mathbb{A}} \sigma\left(s, X_s^n, \mathbb{P}_{X_t^n}, a\right) M^n(ds, da) \right| \\
& \leq \left| \int_0^t \int_{\mathbb{A}} b\left(s, X_s, \mathbb{P}_{X_t}, u\right) \mu_s(da).ds - \int_0^t \int_{\mathbb{A}} b\left(s, X_s, \mathbb{P}_{X_t}, a\right) \delta_{u_s^n}(da) ds \right| \\
& + \left| \int_0^t \int_{\mathbb{A}} b\left(s, X_s, \mathbb{P}_{X_t}, u\right) \delta_{u_s^n}(da).ds - \int_0^t \int_{\mathbb{A}} b\left(s, X_s^n, \mathbb{P}_{X_t}, a\right) \delta_{u_s^n}(da) ds \right| \\
& + \left| \int_0^s \int_{\mathbb{A}} \sigma\left(v, X_v, \mathbb{P}_{X_v}, a\right) M(dv, da) - \int_0^t \int_{\mathbb{A}} \sigma\left(v, X_v, \mathbb{P}_{X_v}, a\right) M^n(dv, da) \right| \\
& + \left| \int_0^s \int_{\mathbb{A}} \sigma\left(v, X_v, \mathbb{P}_{X_v}, a\right) M^n(dv, da) - \int_0^t \int_{\mathbb{A}} \sigma\left(v, X_v^n, \mathbb{P}_{X_v^n}, a\right) M^n(dv, da) \right|
\end{aligned}
$$

Then by using Burkholder-Davis-Gundy inequality for the martingale part and the fact that all the functions in equation (1.8) are Lipschitz continuous, it holds that

$$
E\left(|X_t - X_t^n|^2\right) \leq C \int_0^T E\left(|X_s - X_s^n|^2 + \mathbb{W}_2(\mathbb{P}_{X_s^n}, \mathbb{P}_{X_s})^2\right) dt + K_n,
$$

where C is a nonnegative constant and

$$
\begin{aligned}
K_n = E & \left(\left| \int_0^t \int_{\mathbb{A}} b\left(s, X_s, \mathbb{P}_{X_t}, u\right) \mu_s(da) ds - \int_0^t \int_{\mathbb{A}} b\left(s, X_s, \mathbb{P}_{X_t}, a\right) \delta_{u_s^n}(da) ds \right|^2 \right) \\
+E & \left(\left| \int_0^t \int_{\mathbb{A}} \sigma\left(s, X_s, \mathbb{P}_{X_t}, a\right) M(ds, da) - \int_0^t \int_{\mathbb{A}} \sigma\left(s, X_s, \mathbb{P}_{X_t}, a\right) M^n(ds, da) \right|^2 \right) \\
& = I_n + J_n
\end{aligned}
$$

Using the fact that

$$\mathbb{W}_2(\mathbb{P}_{X_s^n}, \mathbb{P}_{X_s})^2 \leq E\left(|X_s - X_s^n|^2\right),$$

we get

$$E\left(|X_t - X_t^n|^2\right) \leq 2C \int_0^T E\left(|X_s - X_s^n|^2\right) dt + K_n. \tag{1.10}$$

Since the sequence $\left(\delta_{u_t^n}(da)\, dt\right)$ *converges weakly to* $\mu_t(da)\, dt$, $P-a.s.$ and b is bounded and continuous in the control variable, then by applying the Lebesgue dominated convergence theorem we get $\lim\limits_{n\to+\infty} I_n = 0$. On the other hand since σ is bounded and continuous in a, applying (1.9) we get $\lim\limits_{n\to+\infty} J_n = 0$. We conclude by using Gronwall's Lemma.

ii) Let u^n and μ as in i) then

$$\begin{aligned}
|J(u^n) - J(\mu)| \; & \leq E\left[\int_0^T \int_{\mathbb{A}} \left|h(t, X_t^n, \mathbb{P}_{X_t^n}, a) - \, h(t, X_t, \mathbb{P}_{X_t}, a)\right| \delta_{u_t^n}(da)\, dt\right] \\
& + E\left[\left|\int_0^T \int_{\mathbb{A}} h(t, X_t, \mathbb{P}_{X_t}, a)\delta_{u_t^n}(da)\, dt - \int_0^T \int_{\mathbb{A}} h(t, X_t, \mathbb{P}_{X_t}, a)\mu_t(da)\, dt\right|\right] \\
& + E\left[\left|g(X_T^n, \mathbb{P}_{X_T^n}) - g(X_T, \mathbb{P}_{X_T})\right|\right]
\end{aligned}$$

The first assertion implies that the sequence (X_t^n) converges to X_t in probability, then by using the assumptions on the coeffcients h and g and the dominated convergence theorem it is easy to conclude .$\boxdot$

Remark 1.2 *According to the last Proposition, it is clear that the infimum among relaxed controls is equal to the infimum among strict controls, which implies the value functions for the relaxed and strict models are the same.*

Chapitre 2

Stability and prevalence of McKean-Vlasov stochastic differential equations with continuous coefficients

(Joint work with N. Khelfallah)

ABSTRACT. We consider various approximation properties for systems driven by a McKean-Vlasov stochastic differential equations (MVSDEs) with continuous coefficients, for which pathwise uniqueness holds. We prove that the solution of such equations is stable with respect to small perturbation of initial conditions, parameters and driving processes. Moreover, the unique strong solutions may be constructed by an effective approximation procedure. Finally we show that the set of bounded uniformly continuous coefficients for which the corresponding MVSDE have a unique strong solution is a set of second category in the sense of Baire.

Key words : McKean-Vlasov stochastic differential equation – Mean-field - Stability - Strong solution - Pathwise uniqueness - Wasserstein metric - Generic property - Baire space - Generic property.

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2.1 Introduction

McKean-Vlasov stochastic differential equations (MVSDE) have been investigated by McKean [56], for the first time, as the counterpart of Vlasov [68] non linear partial differential equations (PDE) arising in statistical physics. They describe the limiting behaviour of an individual particle evolving within a large system of particles, with weak interaction, as the number of particles tends to infinity. These equations are called non linear SDEs in the sense that the coefficients depend not only on the state variable, but also on its marginal distribution and their solutions are called non linear diffusions. A pedagogical and rigorous treatment of these equations appear in the seminal Saint-Flour course by Sznitman [66].

Since the pioneering work of McKean [56], a huge literature on existence, uniqueness, numerical schemes and propagation of chaos theorems was developped. Existence and uniqueness of strong solutions were obtained under global Lipschitz coefficients in [35, 45, 66] by using the fixed point theorem on the space of continuous functions with values on the space of probability measures, equipped with Wasserstein distance. MVSDEs with non regular coefficients appear naturally in many mean-field models. The so-called mean-field FitzHugh-Nagumo model and the network of Hodgkin-Huxley neurons are typical examples (see [16]). It is clear that if the coefficients are not globally Lipschitz, the Gronwall inequality and its variants fail, so that fixed point theorems are no longer applicable. Let us point out that contrary to Itô's SDEs, regularity assumptions of local nature on the coefficients, such as locally Lipschitz coefficients do not lead to unique (local) strong solutions (see [64] for counterexamples).

It is well known that if the coefficients are Lipschitz continuous, then MVSDE (4.1) has a unique strong solution $X_t(x)$, which is continuous with respect to the initial condition and coefficients. Moreover, the solution may be constructed by means of various numerical schemes (see [8]).

Our purpose in this paper, is to study strong stability properties of the solution of (4.1) under pathwise uniqueness of solutions and merely continuous coefficients. Since the coefficients are only continuous without additional regularity, one cannot expect to apply Gronwall's lemma. Instead of Gronwall's lemma, we use tightness arguments and the famous Skorokhod selection theorem to prove the desired convergence results. Of course we should not expect precise convergence speed as this last property is based on regularity of the coefficients.

The paper is organized as follows. In the second section we prove that the Euler polygonal scheme is convergent provided that there is pathwise uniqueness. This provides us with an effective way to construct strong solutions for MVSDEs. In the third section we prove that the solution is stable under small perturbation of the initial condition and coefficients. Our results generalize similar ones proved for classical Itô SDEs [9, 37, 47]. In the last section, we show that the set of bounded uniformly continuous coefficients for which strong existence and uniqueness hold is a generic property in the sense of Baire. This means that in the sense of Baire category, most of MVSDEs with bounded uniformly continuous coefficients have unique solutions. This last result extends in particular [2, 9, 10] to MVSDEs.

2.2 Assumptions and preliminaries

2.2.1 The Wasserstein distance

Definition 2.1 *Let (M,d) be a metric space, for which every probability measure on M is a Radon measure (a so-called Radon space). Denote $\mathcal{P}_p(M)$ the collection of all probability measures μ on M with finite moment of order p for some x_0 in M, $\int_M d(x_0,x)^p\mu(dx) < +\infty$. Then the $p-$Wasserstein distance between two probability measures μ and ν in $\mathcal{P}_p(M)$ is defined as*

$$W_p(\mu,\nu) = \left(\inf\left\{\int_{M\times M} d(x,y)^p\gamma(dx,dy);\ \gamma\in\Gamma(\mu,\nu)\right\}\right)^{1/p}$$

where $\Gamma(\mu,\nu)$ denotes the collection of all measures on $M\times M$ with marginals μ and ν on the first and second factors respectively. The set $\Gamma(\mu,\nu)$ is also called the set of all couplings of μ and ν

The Wasserstein metric may be equivalently defined by $W_p(\mu,\nu) = \left(\inf E\left[d(X,Y)^p\right]\right)^{1/p}$ where the infimum is taken over all the joint probability distributions of the random variables X and Y with marginals μ and ν.

In the case where the metric space is replaced by the euclidian space $\mathbb{R}^d$, then the p-Wasserstein distance $W_p(\mu,\nu)$ is defined by :

$$W_p(\mu,\nu)^p = \inf_{\gamma\in\Gamma(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p \, d\pi(x,y)^{1/p}$$

or equivalently $W_p(\mu,\nu)^p = \inf \mathbb{E}[\,|X-Y|^p]$.

In particular if X and Y are square integrable random variables, we have $W_2(P_X, P_Y) \leq \mathbb{E}[\,|X-Y|^2]^{1/2}$.

In the literature the Wasserstein metric is restricted to W_2 while W_1 is often called the Kantorovich-Rubinstein distance because of the role it plays in optimal transport.

2.2.2 Assumptions

Let (B_t) a d-dimensional Brownian motion defined on a probablity space $(\Omega, \mathcal{F}, P)$, equipped with a filtration $(\mathcal{F}_t)$, satisfying the usual conditions. Throughout this paper, we consider McKean-Vlasov stochastic differential equation (MVSDE), called also mean-field stochastic differential equation of the form

$$\begin{cases} dX_t = b(t, X_t, \mathbb{P}_{X_t})dt + \sigma(t, X_t, \mathbb{P}_{X_t})dB_t \\ X_0 = x \end{cases} \tag{2.1}$$

For this kind of stochastic differential equations, the drift b and diffusion coefficient σ depend not only on the state process X_t, but also on its marginal distribution $\mathbb{P}_{X_t}$.

Assume that the coefficients satisfy the following conditions.

$(\mathbf{H_1})$ Assume that

$$b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d$$
$$\sigma : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$$

are Borel measurable functions and continuous in (x,μ) uniformly in $t \in [0,T]$.

$(\mathbf{H_2})$ There exist $C > 0$ such that for any $t \in [0,T], x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$|b(t, x, \mu)| \leq C\,(1 + |x| + W_2(\mu, \delta_0)),$$
$$|\sigma(t, x, \mu)| \leq C\,(1 + |x| + W_2(\mu, \delta_0)),$$

where W_2 is the 2-Wasserstein distance and δ_0 is the Dirac measure at 0.

The following theorem states that under global Lipschitz condition, (2.1) admits a unique solution. Its complete proof is given in [66] for a drift depending linearly on the law of X_t that is $b(t, x, \mu) = \int_{\mathbb{R}^d} b'(t, x, y)\mu(dy)$ and a constant diffusion. The general case as (2.1) is treated in [45] Proposition 1.2. The proof is based on a fixed point theorem on the space of continuous functions with values in $\mathcal{P}_2(\mathbb{R}^d)$ endowed with Wasserstein metric. Note that in [35, 45] the authors consider MVSDEs driven by general Lévy process instead of a Brownian motion.

Theorem 2.1 *Assume* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and*

$(\mathbf{H_3})$ *there exist* $L > 0$ *such that for any* $t \in [0, T], x,\ x' \in \mathbb{R}^d$ *and* $\mu,\ \mu' \in \mathcal{P}_2(\mathbb{R}^d)$,

$|b(t, x, \mu) - b(t, x', \mu')| \leq C\,(|x - x'| + W_2(\mu, \mu'))$,

$|\sigma(t, x, \mu) - \sigma(t, x', \mu')| \leq C\,(|x - x'| + W_2(\mu, \mu'))$,

then MVSDE (2.1) admits a unique solution such that $E[\sup_{t\leq T} |X_t|^2] < +\infty$.

Proof. See [45].⊡

Other versions of the MVSDEs, which are particular cases of (2.1) have been considered in the literature.

1) The following MVSDE has been treated in literature

$$\begin{cases} dX_t = b(t, X_t, \int \varphi(y)\mathbb{P}_{X_t}(dy))dt + \sigma(t, X_t, \int \psi(y)\mathbb{P}_{X_t}(dy))dW_t \\ X_0 = x, \end{cases} \tag{2.2}$$

2) MVSDEs studied in the framework of statistical physics take the form

$$\begin{cases} dX_t = \int_{\mathbb{R}^d} b(t, X_t, y)\mathbb{P}_{X_t}(dy)dt + dB_t \\ X_0 = x \end{cases}$$

where $b : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a Borel measurable function such that $b(t,.,.)$ is Lipschitz. This is a particular class of MVSDEs for interacting diffusions, considered by McKean (see [66] for details), where the drift is linear on the probability distribution. It is easy to see that the drift is Lipschitz in the measure variable with respect to Wasserstein metric.
The definition of pathwise uniqueness for equation (2.1) is given by the following.

Definition 2.2 *We say that pathwise uniqueness holds for equation (2.1) if X and X' are two solutions defined on the same probability space $(\Omega, \mathcal{F}, P)$ with common Brownian motion (B), with possibly different filtrations such that $P[X_0 = X_0'] = 1$, then X and X' are indistinguishable.*

Let us recall Kolmogorov's tightness criteia for stochastic processes and Skorokhod selection theorem, which will be extensively used in the sequel.

Lemma 2.1 *(Skorokhod selection theorem [44] page 9) Let (S, ρ) be a complete separable metric space, $P_n, n = 1, 2, ...$ and P be probability measures on $(S, \mathcal{B}(S))$ such that $P_n \underset{n \to +\infty}{\longrightarrow} P$. Then, on a probability space $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}\right)$, we can construct S-valued random variables X_n, $n = 1, 2, ...$, and X such that :*

(i) $P_n = \widehat{P}^{X_n}, n = 1, 2, ...,$ and $P = \widehat{P}^X$.

(ii) X_n converges to X, $\widehat{P}$ almost surely.

Lemma 2.2 *(Skorokhod limit theorem [65]) Let $(\Omega, \mathcal{F}, P)$ be a probability space, (H_n) be uniformly bounded processes and (H^n) be a sequence of Brownian motions defined on the same space such that the stochastic integral $\int_O^T H_s^n dW_s^n$ are well defined for each $n \geq 0$. Assume moreover that*

a) $\lim_{h \to 0} \sup_n \sup_{|s-t|<h} P(|H_s^n - H_t^n| > \varepsilon) = 0$

b) (H_s^n, W_s^n) converges to (H_s^0, W_s^0) in probability.

Then $\int_0^T H_s^n dW_s^n$ converges in probability to $\int_0^T H_s^0 dW_s^0$

Lemma 2.3 (*Kolmogorov criterion for tightness [44] page 18*) *Let $(X_n(t))$, $n = 1, 2, ...,$ be a sequence of d-dimensional continuous processes satisfying the following two conditions :*

(i) There exist positive constants M and γ such that $E\left[|X_n(0)|^\gamma\right] \leq M$ for every $n = 1, 2, ...$

(ii) There exist positive constants α, β, M_k, $k = 1, 2, ...,$ such that :

$E\left[|X_n(t) - X_n(s)|^\alpha\right] \leq M_k |t-s|^{1+\beta}$ for every n and $t, s \in [0, k]$, $(k = 1, 2, ...)$.

Then there exist a subsequence (n_k), a probability space $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}\right)$ and d-dimensional continuous processes $\widehat{X}_{n_k}, k = 1, 2, ...,$ and $\widehat{X}$ defined on it such that

1) The laws of $\widehat{X}_{n_k}$ and X_{n_k} coincide.

2) $\widehat{X}_{n_k}(t)$ converges to $\widehat{X}(t)$ uniformly on every finite time interval $\widehat{P}$ almost surely.

2.3 Construction of strong solutions by approximation

It is well known for classical Itô SDEs [44] as well as for McKeanVlasov SDEs [48] that weak existence and pathwise uniqueness imply the existence and uniqueness of a strong solution. This is a corollary of the famous Yamada-Watanabe theorem (see [44]). In this section we prove that under the pathwise uniqueness, the strong solution, may be constructed by means of an approximation procedure and may be written as a measurable functional of the initial condition and the Brownian motion without appealing to the famous Yamada-Watanabe theorem.

Let (Δ^n) be a sequence of partitions of the interval $[0, T]$ where $\Delta^n : 0 = t_0^n < t_1^n < ..... < t_n^n = T$ such that

$$\lim_{n\to+\infty} \|\Delta^n\| = \lim_{n\to+\infty} \max_i \left(t_{i+1}^n - t_i^n\right) = 0$$

Define the Euler polygonal approximation for equation (2.1) by :

$$X_{\Delta^n}(x,t) = x + \int_0^t b(\phi_{\Delta^n}(s), X_{\Delta^n}, P_{\Delta^n})ds + \int_0^t \sigma(\phi_{\Delta^n}(s), X_{\Delta^n}, P_{X_{\Delta^n}})dB_s$$

where $\phi_{\Delta^n}(s) = t_i$, if $t_i^n \leq s < t_{i+1}^n$ and $\|\Delta^n\| = \max_i \left(t_{i+1}^n - t_i^n\right)$ and $X_{\Delta^n} = X_{\Delta^n}(x, \phi_{\Delta^n}(s))$

Theorem 2.2 *Assume* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$*, then under pathwise uniqueness we have :*
1)$\lim\limits_{n\to 0} E\left[\sup\limits_{t\leq T}|X_{\Delta^n}(x,t)-X(x,t)|^2\right]=0$

2) There exists a measurable functional $F:\mathbb{R}^d\times W_0^d\longrightarrow W^d$ *which is adapted such that the unique solution* X_t *can be written* $X(.)=F(X(0),B(.))$*, where* $W^d=C\left(\mathbb{R}_+,\mathbb{R}^d\right)$ *and* $W_0^d=\left\{w\in C\left(\mathbb{R}_+,\mathbb{R}^d\right):w(0)=0\right\}$ *are equipped with their Borel* $\sigma-$*fields and the filtrations of coordinates.*

Proof. 1) Suppose that the conclusion of our theorem is false, then there exists a sequence(Δ_n) and $\delta\geq 0$ such that

$$\liminf_{n\to\infty} E\left[\sup_{t\leq T}|X_{\Delta_n}(x,t)-X_t|^2\right]\geq\delta. \tag{2.3}$$

Let $\mathcal{C}([0,T])$ be the space of continuous functions equipped with the topology of uniform convergence and $\mathcal{P}_2\left(\mathcal{C}([0,T])\right)$ the space of probability measures equipped with the Wasserstein metric.

Using assumptions $(\mathbf{H_1})$, $(\mathbf{H_2})$ and classical arguments of stochastic calculus, it is easy to see that the sequence $(X_{\Delta_n},X,B,P_{X_{\Delta_n}},P_X)$ satisfies the conditions of Kolmogorov criteria (Lemma 2.6), then it is tight in $\mathcal{C}([0,T])^3\times\mathcal{P}_2\left(\mathcal{C}([0,T])\right)^2$.

Then by Skorokhod limit Theorem (Lemma 2.4), there exist a probability space $\left(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{P}\right)$ and a sequence of stochastic processes $\left(\widehat{X_t^n},\widehat{Y_t^n},\widehat{B_t^n},\widehat{\mu_t^n},\widehat{\nu_t^n}\right)$ defined on it such that :

$i)$ the laws of $(X_{\Delta_n},X,B,P_{X_{\Delta_n}},P_X)$ and $\left(\widehat{X_t^n},\widehat{Y_t^n},\widehat{B_t^n},\widehat{\mu_t^n},\widehat{\nu_t^n}\right)$ coincide for every $n\in\mathbb{N}$.

$ii)$ there exists a subsequence also denoted by $\left(\widehat{X_t^n},\widehat{Y_t^n},\widehat{B_t^n},\widehat{\mu_t^n},\widehat{\nu_t^n}\right)$ converging to $\left(\widehat{X_t},\widehat{Y_t},\widehat{B_t},\widehat{\mu_t},\widehat{\nu_t}\right)$ uniformly on every finite time interval $\widehat{P}-$a.s..

It is clear that $\left(\widehat{B}_t^n,\widehat{\mathcal{F}}_t^n\right)$ and $\left(\widehat{B}_t,\widehat{\mathcal{F}}_t\right)$ are Brownian motions with respect the filtrations $\widehat{\mathcal{F}}_t^n=\sigma\left(\widehat{X_s^n},\widehat{Y_s^n},\widehat{B_s^n};s\leq t\right)$ and $\widehat{\mathcal{F}}_t=\sigma\left(\widehat{X_s},\widehat{Y_s},\widehat{B_s};s\leq t\right)$.

Note that the probability measures do not depend upon the random element ω, then $(P_{X^n},P_X)=\left(\widehat{\mu_t^n},\widehat{\nu_t^n}\right)$ and consequently $\left(\widehat{\mu_t^n},\widehat{\nu_t^n}\right)=\left(P_{\widehat{X_t^n}},P_{\widehat{Y_t^n}}\right)$ and $(\widehat{\mu}_t,\widehat{\nu}_t)=\left(P_{\widehat{X_t}},P_{\widehat{Y_t}}\right)$

According to property $i)$ and the fact that X_{Δ_n} and X_t satisfy equation (2.1) and using the fact that the finite-dimensional distributions coincide, we can easily prove that $\forall n\geq 1$,

$\forall t \geq 0$

$$E\left|\widehat{X_t^n} - x - \int_0^t \sigma\left(\phi_{\Delta_n}(s), \widehat{X_s^n}, P_{\widehat{X_t^n}}\right) d\widehat{B_s^n} - \int_0^t b\left(\phi_{\Delta_n}(s), \widehat{X_s^n}, P_{\widehat{X_t^n}}\right) ds\right|^2 = 0,$$

which means that

$$\widehat{X_t^n} = x + \int_0^t \sigma\left(\phi_{\Delta_n}(s), \widehat{X_s^n}, P_{\widehat{X_t^n}}\right) d\widehat{B_s^n} + \int_0^t b\left(\phi_{\Delta_n}(s), \widehat{X_s^n}, P_{\widehat{X_t^n}}\right) ds.$$

Using similar arguments for $\widehat{Y_t^n}$, we obtain :

$$\widehat{Y_t^n} = x + \int_0^t \sigma\left(s, \widehat{Y_t^n}, P_{\widehat{Y_t^n}}\right) d\widehat{B_s^n} + \int_0^t b\left(s, \widehat{Y_t^n}, P_{\widehat{Y_t^n}}\right) ds$$

Now, by Skorokhod's limit Theorem ([37] Lemma 3.1) and according to $ii)$ and the fact that $\phi_\Delta(s) \to s$, it holds that,

$$\int_0^t \sigma\left(\phi_{\Delta_n}(s), \widehat{X}_s^{n_k}, P_{\widehat{X}_s^{n_k}}\right) d\widehat{B_s^{n_k}} \xrightarrow[k\to\infty]{P} \int_0^t \sigma\left(s, \widehat{X}_s, P_{\widehat{X}_s}\right) d\widehat{B}_s,$$

$$\int_0^t b\left(\phi_{\Delta_n}(s), \widehat{X}_s^{n_k}, P_{\widehat{X}_s^{n_k}}\right) ds \xrightarrow[k\to+\infty]{P} \int_0^t b\left(s, \widehat{X}_s, P_{\widehat{X}_s}\right) ds.$$

We conclude that $\widehat{X_t}$ and $\widehat{Y_t}$ satisfy the same stochastic differential equation (2.1) on the new probability space $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}\right)$, with the same initial condition x and common Brownian motion $\widehat{B_t}$. Therefore, according to the pathwise uniqueness for (2.1) it holds that $\widehat{X} = \widehat{Y}$. By uniform integrability, it holds that :
$\delta \leq \liminf_{n\in\mathbf{N}} E\left[\sup_{t\leq T} |X_{\Delta_n} - X_t|^2\right] \leq \lim \widehat{E}\left[\sup_{t\leq T}\left|\widehat{X_t^{n_k}} - \widehat{Y_t^{n_k}}\right|^2\right] = \widehat{E}\left[\sup_{t\leq T}\left|\widehat{X_t} - \widehat{Y_t}\right|^2\right] = 0$
which contradicts our hypothesis (2.3).

2) Let $(W, \mathcal{B}(W), P^B, B(t))$ be the standard Wiener process and $X_{\Delta_n}(x, ., w)$ be the polygonal approximation . It is clear that the functional $F_{\Delta_n} : \mathbb{R}^d \times W_0^d \longrightarrow W^d$ defined by $F_{\Delta_n}(x, w) = X_{\Delta_n}(x, ., w)$ is measurable. Moreover property 1) and Borel Cantelli lemma imply that $(F_{\Delta_n}(x, w))$ converges uniformly in W^d a.s..

Let $F(x, w) = \lim F_{\Delta_n}(x, w)$, then $F(x, w)$ is measurable and that the unique solution is written as $X(X(0),t) = F(X(0),w)$ which achieves the proof.$\boxdot$

Remark 2.1 *1) Under the same assumptions and using the same proof, we can prove*

$\lim\limits_{n\to 0}\sup\limits_{x\in K} E\left[\sup\limits_{t\leq T}|X_{\Delta^n}(x,t)-X(x,t)|^2\right] = 0$ where K is any compact set in $\mathbb{R}^d$.

Corollary 2.1 *Assume that the coefficients are one dimensional, σ does not depend on the probability measure μ and the coefficients of the MVSDE(2.1) satisfy :*

$(\mathbf{A_1})$ *There exist $C>0$, such that for every $x\in\mathbb{R}$ and $(\mu,\nu)\in\mathcal{P}_1(\mathbb{R})\times\mathcal{P}_1(\mathbb{R})$:*

$|b(t,x,\mu)-b(t,x,\nu)|\leq CW_2(\mu,\nu)$

$(\mathbf{A_2})$ *There exist $K>0$, such that for every $x,y\in\mathbb{R}$, $|\sigma(t,x)-\sigma(t,y)|\leq K|x-y|)$.*

$(\mathbf{A_3})$ *There exists a strictly increasing function $\kappa(u)$ on $[0,+\infty)$ such that $\kappa(0)=0$ and κ is concave satisfying $\int\limits_{0+}\kappa^{-1}(u)du=+\infty$, such that for every $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$ and $\mu\in\mathcal{P}_1(\mathbb{R})$,*

$|b(t,x,\mu)-b(t,y,\mu)|\leq\kappa(|x-y|)$.

Then the conclusion of Theorem 3.1 is valid.

Proof. It is sufficient to show that pathwise uniqueness holds for equation (2.1) (see [8])$\boxdot$

2.4 Stability with respect to initial conditions and coefficients

In this section we will prove that under minimal assumptions on the coefficients and pathwise uniqueness of solutions, the unique solution is continuous with respect the initial condition and coefficients.

We denote by (X_t^x) the unique solution of (2.1) corresponding to the initial condition $X_0^x = x$.

$$\begin{cases} dX_t^x = b(t,X_t^x,\mathbb{P}_{X_t^x})dt + \sigma(t,X_t^x,\mathbb{P}_{X_t^x})dB_t \\ X_0^x = x. \end{cases}$$

Theorem 2.3 *Assume that $b(t,x,\mu)$ and $\sigma(t,x,\mu)$ satisfy $(\mathbf{H_1})$, $(\mathbf{H_2})$. Then if the pathwise uniqueness holds for equation (2.1) then the mapping*

$$\Phi : \mathbb{R}^d \longrightarrow L^2(\Omega, \mathcal{C}([0,T], \mathbb{R}^d))$$

defined by $(\Phi(x)_t) = (X_t^x)$ is continuous.

Proof. Suppose that the conclusion of our theorem is false, then there exists a sequence (x_n) in $\mathbb{R}^d$converging to x and $\delta \geq 0$ such that

$$\liminf_{n\to\infty} E\left[\sup_{t\leq T} |X_t^n - X_t|^2\right] \geq \delta \tag{2.4}$$

where $X_t^n = X_t^{x_n}$ and $X_t = X_t^x$.

Using assumptions $(\mathbf{H_1})$, $(\mathbf{H_2})$ and classical arguments of stochastic calculus, it is easy to see that

$$E\left[|X^n(t) - X^n(s)|^4\right] \leq C(T)|t-s|^2.$$

where $C(T)$ is a constant which does not depend on n. Similar to estimate holds true also for X and the Brownian motion B. Then by Prokhorov's Theorem , the sequence $(X^n, X, P_{X^n}, P_X, B)$ satisfy i) and ii) of Lemma 2.6. therefore this sequence is tight, which implies that it is relatively compact in the topology of weak convergence of probability measures.Hence by Skorokhod selection Theorem (Lemma 2.4), there exists a probability space $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}\right)$ carrying a sequence of stochastic processes $\left(\widehat{X_t^n}, \widehat{Y_t^n}, \widehat{B_t^n}, \widehat{\mu_t^n}, \widehat{\nu_t^n}\right)$ defined on it such that :

$i)$ the laws of $(X^n, X, B, P_{X^n}, P_X)$ and $\left(\widehat{X_t^n}, \widehat{Y_t^n}, \widehat{B_t^n}, \widehat{\mu_t^n}, \widehat{\nu_t^n}\right)$ coincide for every $n \in \mathbb{N}$.

$ii)$ there exists a subsequence also denoted by $\left(\widehat{X_t^n}, \widehat{Y_t^n}, \widehat{B_t^n}, \widehat{\mu_t^n}, \widehat{\nu_t^n}\right)$ converging to $\left(\widehat{X_t}, \widehat{Y_t}, \widehat{B_t}, \widehat{\mu_t}, \widehat{\nu_t}\right)$ uniformly on every finite time interval $\widehat{P}$-a.s., where $\left(\widehat{B_t^n}, \widehat{\mathcal{F}_t^n}\right)$ and $\left(\widehat{B}_t, \widehat{\mathcal{F}}_t\right)$ are Brownian motions with respect the filtrations $\widehat{\mathcal{F}}_t^n = \sigma\left(\widehat{X_s^n}, \widehat{Y_s^n}, \widehat{B_s^n}; s \leq t\right)$ and $\widehat{\mathcal{F}}_t = \sigma\left(\widehat{X_s}, \widehat{Y}_s, \widehat{B_s}; s \leq t\right)$.

Note that the probability measures do not depend upon the random element ω, then $(P_{X^n}, P_X) = \left(\widehat{\mu_t^n}, \widehat{\nu_t^n}\right)$ and consequently $\left(\widehat{\mu_t^n}, \widehat{\nu_t^n}\right) = \left(P_{\widehat{X_t^n}}, P_{\widehat{Y_t^n}}\right)$ and $(\widehat{\mu}_t, \widehat{\nu}_t) = \left(P_{\widehat{X_t}}, P_{\widehat{Y_t}}\right)$

According to property $i)$ and the fact that X_t^n and X_t satisfy equation (2.1) with initial data x_n and x, and using the fact that the finite-dimensional distributions coincide, we can easily prove that $\forall n \geq 1$, $\forall t \geq 0$

$$E\left|\widehat{X_t^n} - x_n - \int_0^t \sigma\left(s, \widehat{X_s^n}, P_{\widehat{X_t^n}}\right) d\widehat{B_s^n} - \int_0^t b\left(s, \widehat{X_s^n}, P_{\widehat{X_t^n}}\right) ds\right|^2 = 0.$$

In other words,

$$\widehat{X_t^n} = x_n + \int_0^t \sigma\left(s, \widehat{X_s^n}, P_{\widehat{X_t^n}}\right) d\widehat{B_s^n} + \int_0^t b\left(s, \widehat{X_s^n}, P_{\widehat{X_t^n}}\right) ds$$

Using similar arguments for $\widehat{Y_t^n}$, we obtain :

$$\widehat{Y_t^n} = x + \int_0^t \sigma\left(s, \widehat{Y_t^n}, P_{\widehat{Y_t^n}}\right) d\widehat{B_s^n} + \int_0^t b\left(s, \widehat{Y_t^n}, P_{\widehat{Y_t^n}}\right) ds$$

Now, by Skorokhod's limit theorem (Lemma 2.5) and according to $ii)$ it holds that,

$$\int_0^t \sigma\left(s, \widehat{X}_s^{n_k}, P_{\widehat{X}_s^{n_k}}\right) d\widehat{B_s^{n_k}} \xrightarrow[k\to\infty]{P} \int_0^t \sigma\left(s, \widehat{X}_s, P_{\widehat{X}_s}\right) d\widehat{B}_s,$$

$$\int_0^t b\left(s, \widehat{X}_s^{n_k}, P_{\widehat{X}_s^{n_k}}\right) ds \xrightarrow[k\to+\infty]{P} \int_0^t b\left(s, \widehat{X}_s, P_{\widehat{X}_s}\right) ds.$$

We conclude that $\widehat{X_t}$ and $\widehat{Y}_t$ satisfy the same stochastic differential equation (2.1) on the new probability space $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}\right)$, with the same initial condition x and common Brownian motion $\widehat{B_t}$

$$\widehat{X_t} = x + \int_0^t \sigma\left(s, \widehat{X_s}, P_{\widehat{X_t}}\right) d\widehat{B_s} + \int_0^t b\left(s, \widehat{X_s}, P_{\widehat{X_t}}\right) ds$$

and

$$\widehat{Y}_t = x + \int_0^t \sigma\left(s, \widehat{Y}_t, P_{\widehat{Y}_t}\right) d\widehat{B_s} + \int_0^t b\left(s, \widehat{Y}_t, P_{\widehat{Y}_t}\right) ds.$$

According to the pathwise uniqueness for (2.1) it holds that $\widehat{X} = \widehat{Y}$.

By uniform integrability, it holds that :
$\delta \leq \liminf_{n\in\mathbf{N}} E\left[\sup_{t\leq T} |X_t^n - X_t|^2\right] \leq \lim \widehat{E}\left[\sup_{t\leq T}\left|\widehat{X_t^{n_k}} - \widehat{Y_t^{n_k}}\right|^2\right] = \widehat{E}\left[\sup_{t\leq T}\left|\widehat{X_t} - \widehat{Y}_t\right|^2\right] = 0$
which contradicts our hypothesis (2.4).$\boxdot$

Using the same techniques we can prove the continuity of the solution of MVSDE with respect to a parameter. In particular the solution is continunous with respect to the coefficients. Let us consider a sequence of functions and consider the MVSDE.

$$\begin{cases} dX_t^n = \sigma_n\left(t, X_t^n, P_{X_t^n}\right) dB_t + b_n\left(t, X_t^n, P_{X_t^n}\right) dt \\ X^n(0) = x_n. \end{cases} \tag{2.5}$$

Theorem 2.4 *Suppose that $\sigma_n(t,x,\mu)$ and $b_n(t,x,\mu)$ are continuous functions . Further suppose that for each $T>0$, and each compact set K there exists $L>0$ such that*

i) $\sup\limits_{t\leq T}\left(\left|\sigma_n(t,x,\mu)\right| + \left|b_n(t,x,\mu)\right|\right) \leq L\ \ (1+|x|)$ uniformly in n,
ii) $\lim\limits_{n\to+\infty}\ \sup\limits_{x\in K}\sup\limits_{t\leq T}\left(\left|\sigma_n(t,x,\mu)-\sigma(t,x,\mu)\right| + \left|b_n(t,x,\mu)-b(t,x,\mu)\right|\right) = 0$,
iii) $\lim_{n\to+\infty} x_n = x$.

If the pathwise uniqueness holds for equation (2.1), then :

$$\sup_{x\in K} E\left[\sup_{t\leq T}\left|X_t^n - X_t\right|^2\right] = 0 \text{ , for every } T\geq 0 \text{ .}$$

Proof. Similar to the proof of Theorem 4.1. ⊡

Corollary 2.2 *Under the assumptions of Corollary 3.3, the conclusions of Theorem 4.1 and Theorem 4.2 are valid.*

Proof. Under the conditions of Corollary 3.3, the pathwise uniqueness holds for equation (2.1) (see [8]), then the conditions of Theorem 4.1. are satisfied and the conclusion holds.⊡

2.5 Existence and uniqueness is a generic property

We know that under globally Lipschitz coefficients equation (2.1) has a unique strong solution (see [35, 45, 66]). A huge literature has been produced to improve the conditions under which pathwise uniquenes holds. Moreover the continuity of the coefficients is not sufficient for the uniqueness. The objective to identify completely the set of coefficients, under which there is a unique strong solution seems to be out of reach, even for ordinary differential equations. In this section we are interested in qualitative properties of the set of coefficients for which existence and uniqueness of solutions hold. In fact we prove that "most" of the MVSDEs with bounded uniformly continuous coefficients enjoy the property of existence and uniqueness. The expression "most" should be understood in the sense of topology and is similar to the measure theoritic concept of a set whose complement is a negligible set. More precisely, we prove that in the sense of Baire, the set of coefficients (b, σ) for which existence and uniqueness of a strong solution is a residual set in the Baire space of all bounded uniformly continuous functions.

Prevalence properties for ordinary differential equations were first considered by Orlicz [61] and Lasota-Yorke [50]. Similar properties for Itô stochastic differential equations have been inverstigated in [2, 9, 10, 42].

Let us recall some facts about Baire spaces.

Definition 2.3 *A Baire space X is a topological space in which the union of every countable collection of closed sets with empty interior has empty interior.*

This definition is equivalent to each of the following conditions.

a) Every intersection of countably many dense open sets is dense.

b) The interior of every union of countably many closed nowhere dense sets is empty.

Remark 2.2 *By the Baire category theorem, we know that a complete metric space is a Baire space.*

Definition 2.4 *1) A subset of a topological space X is called nowhere dense in X, if the interior of its closure is empty*

2) A subset is of first category in the sense of Baire (or meager in X), if it is a union of countably many nowhere dense subsets.

3) A subset is of second category or nonmeager in X, if it is not of first category in X.

Remark 2.3 *1) The definition for a Baire space can then be stated as follows : a topological space X is a Baire space if every non-empty open set is of second category in X.*

2) In the literature, a subset of second category is also called a residual subset.

Definition 2.5 *A property P is generic in the Baire space $\mathcal{X}$ if P holds is satisfied for each element in $\mathcal{X}-\mathcal{N}$, where $\mathcal{N}$ is a set of first category in the Baire space $\mathcal{X}$.*

Let us introduce the appropriate Baire space.

Let $\mathcal{C}_1$be the set of bounded uniformly continuous functions $b:\mathbb{R}_+\times\mathbb{R}^d\times\mathcal{P}_2\left(\mathbb{R}^d\right)\longrightarrow\mathbb{R}^d$. Define the metric ρ_1 on $\mathcal{C}_1$ as follows :

$$\rho_1\left(b_1,b_2\right)=\sup_{(t,x,\mu)\in\mathbb{R}_+\times\mathbb{R}^d\times\mathcal{P}_2\left(\mathbb{R}^d\right)}\left|b_1(t,x,\mu)-b_2(t,x,\mu)\right|$$

Note that the metric ρ_1 is compatible with the topology of uniform convergence on $\mathbb{R}_+\times\mathbb{R}^d\times\mathcal{P}_2\left(\mathbb{R}^d\right)$.

Let $\mathcal{C}_2$ be the set of bounded unifomly continuous functions $\sigma:\mathbb{R}_+\times\mathbb{R}^d\times\mathcal{P}_2\left(\mathbb{R}^d\right)\longrightarrow\mathbb{R}^d\otimes\mathbb{R}^d$ endowed with the corresponding metric ρ_2 :

$$\rho_2\left(\sigma_1,\sigma_2\right)=\sup_{(t,x,\mu)\in\mathbb{R}_+\times\mathbb{R}^d\times\mathcal{P}_2\left(\mathbb{R}^d\right)}\left|\sigma_1(t,x,\mu)-\sigma_2(t,x,\mu)\right|$$

It is clear that since $\mathbb{R}_+\times\mathbb{R}^d\times\mathcal{P}_2\left(\mathbb{R}^d\right)$ is a complete metric space, then $\mathfrak{R}=\mathcal{C}_1\times\mathcal{C}_2$ endowed with the metric λ is a complete metric space also, where $\lambda((b_1,\sigma_1),(b_2,\sigma_2))=\rho\left(b_1,b_2\right)+\rho\left(\sigma_1,\sigma_2\right)$.

Remark 2.4 *Note that for ordinary or Itô stochastic differential equations, the suitable Baire space is the space of bounded continous functions. The space of continuous functions contains a dense subset formed of all locally Lipschitz functions for which there is uniqueness of solutions for Itô SDEs. This property is no more valid for MVSDEs as the uniqueness of solutions may fail for locally Lipschitz coefficients (see [64]). Instead of bounded continuous functions we consider bounded uniformly continuous functions. These functions are approximated by globally Lipschitz functions for which we have existence and uniquness. The* *fact that the coefficients depend on the marginal distribution of the unknown process is not suitable for localization techniques.*

For (b,σ) in $\mathfrak{R}$, let $E(x,b,\sigma)$ stands for MVSDE (2.1) corresponding to coefficients b,σ and initial data x.

$$\mathbf{M}^2 = \left\{ \xi : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}^d,\, F_t^B - \text{adapted, continuous with } E\left[\sup_{t\leq T} |\xi_t|^2\right] < +\infty \right\}$$

Define a metric on $\mathbf{M}^2$ by :

$$d\left(\xi_1, \xi_2\right) = \left(E \sup_{0\leq t\leq T} \left|\xi_t^1 - \xi_t^2\right|^2 \right)^{\frac{1}{2}}$$

By using Borel-Cantelli lemma, it is easy to see that $(\mathbf{M}^2, d)$ is a complete metric space.

It is clear that a strong solution (ξ_t) of equations (2.1) is an element of the metrix space $(\mathbf{M}^2, d)$.

Let $\mathcal{L}$ be the subset of $\mathfrak{R}$ consisting of functions $h(t,x,\mu)$ which are Lipschitz in their arguments, that is :

$$|b(t,x,\mu) - b(t,x',\mu')| \leq C\left(|x-x'| + W_2(\mu,\mu'\right),$$
$$|\sigma(t,x,\mu) - \sigma(t,x',\mu')| \leq C\left(|x-x'| + W_2(\mu,\mu'\right)$$

Proposition 2.1 *Every bounded uniformly continuous function in a metric space is a uniform limit of a sequence of globally Lipschitz functions.*

Proof. See [41] Theorem 6.8$\boxdot$

The last proposition states that the subset $\mathcal{L}$ of globally Lipschitz functions is dense in the Baire space $\mathcal{R}$.

2.5.1 The oscillation function

Let us define the oscillation function, which was first introduced by Lasota-Yorke [50] in the case of ordinary differential equations and partial differential equations and then used by [10, 42] for Itô SDEs.

Let $x \in \mathbb{R}^d$ and $(b,\sigma) \in \mathfrak{R}$, let $\xi(x,b,\sigma)$ the solution of equation $E(x,b,\sigma)$.

Define the oscillation function as follows

$$D_1(x,b,\sigma):\mathbb{R}^d\times\mathcal{R}\longrightarrow\mathbb{R}_+$$

$$D_1(x,b,\sigma)=\lim_{\delta\to0}\sup\{d(\xi(x,b_1,\sigma_1),\xi(x,b_2,\sigma_2);\;\;(b_i,\sigma_i)\in\mathcal{L}\text{ and }\lambda((b,\sigma),(b_i,\sigma_i))<\delta,\;i=1,2\}$$

Proposition 2.2 *Let $x\in\mathbb{R}^d$ and (b,σ) are Lipschitz coefficients, that is $(b,\sigma)\in\mathcal{L}$, then* $D_1(x,b,\sigma)=0$.

Proof.

We know that if $(b,\sigma)\in\mathcal{L}$ then equation (2.1) has a unique strong solution.

For each $i=1,2$, let (X_t^i) be a solution of (2.1) coerresponding to (b_i,σ_i), then

$$\begin{aligned}|X_t^1-X_t^2|^2\leq{}&3(\int_0^t|b_1(s,X_s^1,\mathbb{P}_{X_s^1})-b_1(s,X_s^2,\mathbb{P}_{X_s^2})|ds)^2\\&+3(\int_0^t|b_1(s,X_s^2,\mathbb{P}_{X_s^2})-b_2(s,X_s^2,\mathbb{P}_{X_s^2}))|ds)^2\\&+3\left|\int_0^t\left(\sigma_1(s,X_s^1,\mathbb{P}_{X_s^1})-\sigma_1(s,X_s^2,\mathbb{P}_{X_s^2})\right)dB_s\right|^2\\&+3\left|\int_0^t\left(\sigma_1(s,X_s^2,\mathbb{P}_{X_s^2})-\sigma_2(s,X_s^2,\mathbb{P}_{X_s^2})\right)dB_s\right|^2.\end{aligned}$$

By using the Lipschitz continuity and Burkholder Davis Gundy inequality, it holds that

$$\begin{aligned}E\left[\sup_{t\leq T}|X_t^1-X_t^2|^2\right]\leq{}&3(T+C_2)L^2\int_0^t\left(E\left[\sup_{s\leq t}|X_s^1-X_s^2|^2\right]+W_2(\mathbb{P}_{X_s^1},\mathbb{P}_{X_s^2})^2\right)ds\\&+6(T+C_2)E[\int_0^t|b_1(s,X_s^2,\mathbb{P}_{X_s^2})-b(s,X_s^2,\mathbb{P}_{X_s^2}|^2ds]\\&+6(T+C_2)E[\int_0^t\int_0^t|\sigma_1(s,X_s^2,\mathbb{P}_{X_s^2})-\sigma(s,X_s^2,\mathbb{P}_{X_s^2}|^2ds]\\&\leq6(T+C_2)\int_0^tE\left[\sup_{s\leq t}|X_s^1-X_s^2|^2\right]ds+K,\end{aligned}$$

such that

$$K=3(T+C_2)E[\int_0^T|b_1-b|^2(s,X_s^2,\mathbb{P}_{X_s^2})+|\sigma_1-\sigma|^2(s,X_s^2,\mathbb{P}_{X_s^2})ds].$$

An application of Gronwall lemma allows us to get

$$E\left[\sup_{t\leq T}|X_t^1 - X_t^2|^2\right] \leq C\delta^2.$$

where C is some constantwhcih implies that $D_1(x,b,\sigma) = 0$.⊡

Proposition 2.3 *The oscillation function D is upper semicontinuous function at each point of the set $\mathbb{R}^d \times \mathcal{L}$.*

Proof.

Let (x_n, b_n, σ_n) be a sequence in $\mathbb{R}^d \times \mathcal{R}$ converging to a limit $(x,b,\sigma) \in \mathbb{R}^d \times \mathcal{L}$. D is upper semicontinuous if $\lim_{n\to+\infty} D_1\left(x_n, b_n, \sigma_n\right) = 0$. Suppose that the last statement is false. Then according to the definition of the function D, there exists $\varepsilon > 0$ and a subsequence still denoted by $\{n\}$ (to avoid heavy notations) and functions (b_n^i, σ_n^i) in $\mathcal{L}$ such that :

$$\text{(i) } \lambda(\left(b_n, \sigma_n\right), \left(b_n^i, \sigma_n^i\right)) < 1/2^n$$
$$\text{(ii) } d(\xi(x_n, b_n^1, \sigma_n^1), \xi(x_n, b_n^2, \sigma_n^2)) > \varepsilon/2$$

Thus according to Theorem 4.1 on the continuous dependence with respect to initial condition and coefficients, and property (i) it holds that :

$$\lim_{n\to+\infty} d(\xi(x_n, b_n^1, \sigma_n^1), \xi(x_n, b_n^2, \sigma_n^2)) = 0.$$

But this contradicts the property (ii), then D is upper semicontinuous.⊡

Proposition 2.4 *Let (x,b,σ) be in $\mathbb{R}^d \times \mathcal{R}$ such that $D_1\left(x,b,\sigma\right) = 0$, then there exists at least one strong solution to MVSDE (2.1).*

Proof. Similar to [10] Prop. 1.4 or Proposition 5 in [42].⊡

2.5.2 Existence and uniqueness of solutions is a generic propery

The main result of this section is the following.

Theorem 2.5 *The subset $\mathcal{U}$ consisting of those (σ, b) for which existence and uniqueness of a strong solution holds for equation (2.1) contains a set of second category in the Baire space $\mathfrak{R}$.*

Proof. It is clear from Proposition 5.10, that if for some (σ, b) in $\mathfrak{R}$, equation (2.1) has at least one strong solution then $D_1(x, b, \sigma) = 0$. Then the set of couples (σ, b) in $\mathfrak{R}$, for which existence of strong solution holds, contains the set

$$\mathcal{A} = \{(\sigma, b) \in \mathfrak{R}; D_1(x, b, \sigma) = 0\}.$$

If we denote

$$\mathcal{A}_n = \{(\sigma, b) \in \mathfrak{R}; D_1(x, b, \sigma) < 1/n\}$$

then $\mathcal{A} = \bigcap\limits_{n=1}^{+\infty} \mathcal{A}_n$.

Let $(b, \sigma) \in \mathcal{L}$, then according to Proposition 5.8 we have $D_1(x, b, \sigma) = 0$. Therefore $\mathcal{L} \subset \mathcal{A}_n$ and then by Proposition 5.7, $\mathcal{A}_n$ contains a dense open subset of the Baire space $(\mathfrak{R}, \lambda)$. Therefore $\mathcal{A}$ contains an intersection of open dense subsets, then $\mathcal{A}$ is a residual subset in $\mathfrak{R}$. If $(b, \sigma) \in \mathcal{A}$, equation MVSDE (2.1) enjoys the property of existence of a solution. To obtain the property of uniqueness let us introduce the finction $D_2 : \mathcal{A} \longrightarrow [0, +\infty[$ defined by

$$D_2((b, \sigma)) = \sup\{d(\xi_1, \xi_2); \xi_i \in \mathbf{S}^2 \text{ and } \xi_i \text{ is a strong solution of } E(x, b, \sigma)\}$$

and

$$\mathcal{B}_n = \{(\sigma, b) \in \mathcal{A}; D_2(x, b, \sigma) < 1/n\}$$

Let $\mathcal{B} = \bigcap\limits_{n=1}^{+\infty} \mathcal{B}_n$

Let us note that if (b, σ) are Lipschitz functions then equation (2.1) admits a unique strong solution. Therefore if $(b, \sigma) \in \mathcal{L}$, then $D_2(x, b, \sigma) = 0$. This implies in particular that $\mathcal{B}_n$ contains the intersection of $\mathcal{A}$ and a dense open subset in $\mathcal{R}$, namely $\mathcal{L}$. Therefore $\mathcal{B}$ contains an intersection of open dense subsets in the Baire space $(\mathfrak{R}, \lambda)$. This means that $\mathcal{B}$ is a residual subset in $(\mathfrak{R}, \lambda)$. $\boxdot$

Remark 2.5 *By using similar techniques it is not difficult to prove that the set of coefficients* (b, σ) *for which the Euler polygonal scheme and the Picard scheme for MVSDE converge, is a residual set in the Baire space of all bounded uniformly continuous functions* $(\mathfrak{R}, \lambda)$.

Chapitre 3

On the convergence of Carathéodory numerical scheme for Mckean-Vlasov equations

ABSTRACT. We study the strong convergence of the Carathéodory numerical scheme for a class of nonlinear McKean-Vlasov stochastic differential equations (MVSDE). We prove, under Lipschitz assumptions, the convergence of the approximate solutions to the unique solution of the MVSDE. Moreover, we extend the above result to continuous coefficients, provided that pathwise uniqueness holds for the corresponding MVSDE. The proof is based on weak convergence techniques and the Skorokhod embedding theorem. In particular, this general result allows us to construct the unique strong solution of a MVSDE, by using the Carathéodory numerical scheme. We give examples of MVSDEs with non Lipschitz coefficients, for which our result holds.

Keywords : McKean-Vlasov equation - Mean-field equation - Carathéodory numerical scheme - Wasserstein distance - Delay equation - Tightness - Pathwise uniqueness - Strong solution.

2010 Mathematics Subject Classification. 60H10, 65C30, 65L20, 93E15.

3.1 Introduction

The purpose of this paper is to study the convergence of the Carathéodory numerical scheme, for a class of nonlinear McKean-Vlasov stochastic differential equations (MVSDEs). This type of equation describes the evolution of a single particle, interacting weakly with similar particles through the empirical measure of their states. When the number of particles tends to infinity, the whole set of particles converges to a continuum of independent particles, satisfying the same equation, called mean-field equation or McKean-Vlasov equation. The mean-field equation has coefficients depending on the state process as well as on its distribution. For a good introduction, one can refer to the excellent Snitzman's course [66]. In the last decade, there has been a renewed interest in the study of MVSDEs, because of their intimate relationship with the so-called mean-field games [20]. Existence and uniqueness of strong solutions were obtained under global Lipschitz coefficients in [45, 66], by using the fixed point theorem. MVSDEs with non Lipschitz coefficients turned out to be important in real world applications. A typical example is the so-called stochastic mean-field FitzHugh - Nagumo model or the network of Hodgkin Huxley neurons, appearing in biology or physics. We refer to [11] for a discussion of this kind of models. These equations have motivated many authors to improve the classical Lipschitz case, see [4, 8, 21, 22, 28, 38, 59].

As it is well know, explicit or closed forms of the solutions of such equations, are not available, except in rare situations. Therefore, to be able to use MVSDEs as efficient modelling tools, one needs numerical methods to approximate their solutions. There is a wide literature on approximation schemes for the solutions of stochastic differential equations (SDEs), such as the well known Picard successive approximation scheme, Euler procedure and Cauchy Maruyama method for SDEs. Numerical methods for MVSDEs [27, 34, 33] is an interesting growing topic, motivated by various applications such as optimal control and games, mathematical finance and large networks.

Our aim in this paper is to study the convergence of the Carathéodory numerical scheme, for a class of nonlinear MVSDEs. To the best of our knowledge, this is the first paper, dealing with such a numerical scheme in the context of MVSDEs. This approximate scheme is defined by a sequence of solutions of MVSDEs with small delays. It has been introduced for the first time in [25], for ordinary differential equations. The extension to SDEs of the Itô type has been performed in [12]. In [54, 55], the author considered the convergence of this scheme for SDEs with variable delays. The convergence of the Carathédory scheme and the existence of a solution for functional SDEs have been investigated in [67]. In [13] the authors studied the existence and uniqueness, via the Carathéodory scheme, for perturbed SDEs with reflecting boundary. The convergence of this scheme for SDEs with non Lipschitz coefficients has been investigated in [31] by using weak convergence techniques. In [32], the author has treated the case of SDEs driven by a G-Brownian motion, which was introduced by S. Peng [62] as the natural stochastic process associated with some non linear heat equation. Note that this kind of process has found natural relationship with the concept of risk mesaure and model uncertainty in mathematical finance. The important feature about the Carathéodory iteration scheme, is that each stochastic delay equation can be solved explicitly by successive stochastic integrations, over intervals of small length. The advantage of this numerical scheme is that unlike the Picard scheme, to calculate X^k, we do not need to compute the preceding approximate solutions $X^1X^2, ...X^{k-1}$.

Motivated by the above mentioned papers, we prove two results on the convergence of the Carathéodory scheme. In our first result, namely Theorem 3.1, we prove that, under Lipschitz conditions, the Carathédory scheme converges to the unique solution of our MVSDE. This result extends that in ([12]) to MVSDEs and provides an alternative proof of the existence and uniqueness theorem. In our second main result, namely Theorem 4.1, we prove the convergence of the Carathéodory scheme for MVSDEs with pathwise uniqueness, whose coefficients are not Lipschitz. As we have no additional information on the modulus of continuity of the coefficients, Gronwall's inequality cannot be used in our situation. To overcome this serious drawback, we use probabilistic arguments such as the tightness of the processes under consideration and the Skorokhod embedding theorem. Moreover we use a deep characterization of the convergence in probability of sequences of random variables, in terms of weak convergence. Unlike the Picard scheme, which is valid under rather strong assumptions of the coefficients, the Carathéodory scheme converges under any condition ensuring pathwise uniqueness. To show the usefulness of our result, we provide examples of MVSDEs with non Lipschitz coefficients, for which the Carathéodory numerical scheme converges, while there is no evidence for the convergence of the Picard scheme.

3.2 Preliminaries

3.2.1 Assumptions

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, equipped with a filtration $(\mathcal{F}_t)$ and (B_t) an $(\mathcal{F}_t, P)-$Brownian motion with values in $\mathbb{R}^d$. The object of study is a McKean-Vlasov stochastic differential equation (MVSDE)

$$\begin{cases} dX_t = b(t, X_t, \mathbb{P}_{X_t})ds + \sigma(t, X_t, \mathbb{P}_{X_t})dB_s \\ X_0 = x, \end{cases} \tag{3.1}$$

whose drift and diffusion coefficient depend not only on the state process but also on its law. Let $\mathcal{P}_2(\mathbb{R}^d)$ be the space of probability measures μ such that $\int_{\mathbb{R}^d} |x|^2 \mu(dx) < +\infty$, equipped with the Wasserstein metric

$$W_2(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} [\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\pi(x,y)]^{1/2},$$

where $\Pi(\mu, \nu)$ denotes the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$, such that $\pi(A \times \mathbb{R}^d) = \mu(A)$ and $\pi(\mathbb{R}^d \times A) = \nu(A)$. We consider the following hypothesis.

$(\mathbf{H_1})$ We assume that

$$b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d$$
$$\sigma : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d,$$

are Borel measurable, continuous functions in (x, μ) with linear growth. There exist $C > 0$ such that for every $(t, x, \mu) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$:

$$|b(t,x,\mu)| + |\sigma(t,x,\mu)| \leq C\left(1 + |x| + W_2(\mu,\delta_0)\right),$$

where δ_0 denotes the Dirac measure at the point 0.

($\mathbf{H_2}$) There exists $L > 0$ s.t. for any $t \in [0,T]$, x, $y \in \mathbb{R}^d$ and μ, $\nu \in \mathcal{P}_2(\mathbb{R}^d)$:

$$|b(t,x,\mu) - b(t,y,\nu)| + |\sigma(t,x,\mu) - \sigma(t,y,\nu)| \leq L(|x-y| + W_2(\mu,\nu)).$$

Theorem 3.1 *Under* ($\mathbf{H}_1$) *and* ($\mathbf{H}_2$), *for each fixed* $X_0 \in \mathbb{R}^d$, *the MVSDE (4.1) has a unique strong solution* $P-$*almost surely continuous with finite second order moment.*

Proof. See [45] Proposition 1.2 ⊡

Remark 3.1 *Using assumption* ($\mathbf{H}_1$) *it is not difficult to show that* $E[|X_t - X_s|^4] \leq M\,|t-s|^2$ *for* $s,t \in [0,T]$ *and* M *a constant depending on* T. *By Kolmogorov continuity criterion ([44] Corollary 4.4, page 20) there exists a continuous version of* (X_t). *Moreover* $t \longrightarrow \mathbb{P}_{X_t}$ *is continuous from* $[0,+\infty)$ *into* $\mathcal{P}_2(\mathbb{R}^d)$ *equipped with weak convergence topology (which is compatible with the Wasserstein metric). This is equivalent to verify that for every bounded continous function* $f : \mathbb{R}^d \longrightarrow \mathbb{R}$, *the function* $t \longrightarrow \int_{\mathbb{R}^d} f(y)\mathbb{P}_{X_t}(dy) = E\left(f(X_t)\right)$ *is continuous. But this results from the Lebesgue dominated convergence theorem.*

3.2.2 The Carathéodory numerical scheme

Let us define the Carathéodory approximation scheme. This is defined by the sequence of stochastic processes given by the following delayed MVSDEs

$$\begin{cases} dX_t^n = b(t, X^n_{t-\frac{1}{n}}, \mathbb{P}_{X^n_{t-\frac{1}{n}}})dt + \sigma(t, X^n_{t-\frac{1}{n}}, \mathbb{P}_{X^n_{t-\frac{1}{n}}})dB_t, \quad if \quad t > 0 \\ X_t^n = x, \quad \text{if} \quad -\frac{1}{n} \leq t \leq 0. \end{cases} \tag{3.2}$$

It is clear that for each fixed integer n, X^n is well defined as a continuous square integrable semi-martingale, which may be constructed recursively over the intervals $[0,\frac{1}{n})$, $[\frac{1}{n},\frac{2}{n})$, $[\frac{2}{n},\frac{3}{n})$, ..., Note that if $-\frac{1}{n} \leq t \leq 0$ then $X_t^n = x$, therefore for $0 \leq t \leq \frac{1}{n}$

$$X_t^n = x + \int_0^t b(s, X^n_{s-\frac{1}{n}}, \mathbb{P}_{X^n_{s-\frac{1}{n}}})dt + \int_0^t \sigma(s, X^n_{s-\frac{1}{n}}, \mathbb{P}_{X^n_{s-\frac{1}{n}}})dB_s.$$

But for $0 \leq s \leq \frac{1}{n}$ we have $-\frac{1}{n} \leq s - \frac{1}{n} \leq 0$ therefore $X^n_{s-\frac{1}{n}} = x$ and

$$X^n_t = x + \int_0^t b(s, x, \mathbb{P}_x)dt + \int_0^t \sigma(s, x, \mathbb{P}_x)dB_s.$$

According to assumption $(\mathbf{H_1})$, X^n_t is well defined on the interval $[0, \frac{1}{n}]$ as the sum of a Lebesgue integral and an Itô stochastic integral.

We can continue to define X^n_t by step-wise iterated Lebesgue and Itô stochastic integration over intervals $[0, \frac{1}{n}), [\frac{1}{n}, \frac{2}{n}), [\frac{2}{n}, \frac{3}{n}),$ .....etc.

According to 1) in Lemma 3.1, (X^n_t) is an $\mathcal{F}_t$-adapted square integrable Itô process. In addition, by using the same techniques as in 2) Lemma 3.1, it is clear that X^n_t satisfies the Kolmogorov continuity criterion, which implies the existence of a continuous version of X^n_t on $\left[-\frac{1}{n}, +\infty\right)$. By using the same arguments as in Remark 2.2, it is immediate that $t \longrightarrow \mathbb{P}_{X^n_{t-\frac{1}{n}}}$ is continuous from $[0, +\infty)$ into $\mathcal{P}_2(\mathbb{R}^d)$.

3.3 The Carathéodory scheme for Lipschitz MVSDEs

To prove the main result of this section we need the following technical Lemma.

Lemma 3.1 *let (X^n) defined by (3.2) and $0 < T < \infty$, then the following estimates hold.*

1) *There exists M a positive constant depending only on T and x, such that, for every $n \geq 1$, $E[\sup_{-1/n \leq s \leq T} |X^n_s|^2] \leq M$.*
2) *There exists a positive constant N depending only on T, such that, For every $-\frac{1}{n} \leq s, t \leq T$ and $n \geq 1$, $E[|X^n_t - X^n_s|^2] \leq N\,|t - s|$.*

Proof. 1) Let us fix $T > 0$ and $t \leq T$, for each integer $k > 0$ define the stopping time

$$T_k = \inf \left\{ t \geq 0; |X^n_t| > k \right\}.$$

$$X^n_{t\wedge T_k} = X_0 + \int_0^{t\wedge T_k} \sigma(s, X^n_{s-\frac{1}{n}}, \mathbb{P}_{X^n_{s-\frac{1}{n}}})dB_s + \int_0^{t\wedge T_k} b(s, X^n_{s-\frac{1}{n}}, \mathbb{P}_{X^n_{s-\frac{1}{n}}})ds.$$

By using the inequality $|a+b+c|^2 \leq 3\left(|a|^2 + |b|^2 + |c|^2\right)$ we get

$$\left|X^n_{t\wedge T_k}\right|^2 \leq 3\left(|X_0|^2 + \left|\int_0^{t\wedge T_k} b(s, X^n_{s-\frac{1}{n}}, \mathbb{P}_{X^n_{s-\frac{1}{n}}})ds\right|^2 + \left|\int_0^{t\wedge T_k} \sigma(r, X^n_{s-\frac{1}{n}}, \mathbb{P}_{X^n_{s-\frac{1}{n}}})dB_s\right|^2\right).$$

Thus we have

$$\begin{aligned} E[\sup_{0\leq s\leq t\wedge T_k} |X^n_s|^2] &\leq 3E\left(|X_0|^2\right) + 3E\left[\sup_{0\leq s\leq t\wedge T_k}\left|\int_0^s \sigma(r, X^n_{r-\frac{1}{n}}, \mathbb{P}_{X^n_{r-\frac{1}{n}}})dB_r\right|^2\right] \\ &\quad + 3E\left[\sup_{0\leq s\leq t\wedge T_k}\left|\int_0^s b(r, X^n_{r-\frac{1}{n}}, \mathbb{P}_{X^n_{r-\frac{1}{n}}})dr\right|^2\right]. \end{aligned}$$

By using the linear growth condition and assumption $(\mathbf{H_1})$ we have

$$E\left[\sup_{0\leq s\leq t\wedge T_k}\left|\int_0^s b(r, X^n_{r-\frac{1}{n}}, \mathbb{P}_{X^n_{r-\frac{1}{n}}})dr\right|^2\right] \leq 3C^2TE\left(\int_0^{t\wedge T_k}\left(1 + \left|X^n_{s-\frac{1}{n}}\right|^2 + W_2^2(\mathbb{P}_{X^n_{s-\frac{1}{n}}}, \delta_0)\right)ds\right).$$

Since $W_2^2(\mathbb{P}_{X^n_{s-\frac{1}{n}}}, \delta_0) \leq E[|X^n_{s-\frac{1}{n}} - 0|^2] \leq E[|X^n_{s-\frac{1}{n}}|^2]$ then

$$E\left[\sup_{0\leq s\leq t\wedge T_k}\left|\int_0^s b(r, X^n_{r-\frac{1}{n}}, \mathbb{P}_{X^n_{r-\frac{1}{n}}})dr\right|^2\right] \leq 3C^2TE\left[\int_0^{t\wedge T_k}\left(1 + 2\left|X^n_{s-\frac{1}{n}}\right|^2\right)ds\right].$$

Again by using assumption $(\mathbf{H_1})$ along with Doob's inequality we get

$$E\left[\sup_{0\leq s\leq t\wedge T_k}\left|\int_0^s \sigma(r, X^n_{r-\frac{1}{n}}, \mathbb{P}_{X^n_{r-\frac{1}{n}}})dr\right|^2\right] \leq 12C^2E\left[\int_0^{t\wedge T_k}\left(1 + 2\left|X^n_{s-\frac{1}{n}}\right|^2\right)ds\right].$$

Therefore

$$E[\sup_{0\leq s\leq T\wedge T_k} |X^n_s|^2] \leq 3(|X_0|^2 + 6C^2T + 24C^2)E\int_0^T (1 + E[\sup_{0\leq r\leq s\wedge T_k} |X^n_r|^2])ds.$$

We may now apply Gronwall's lemma to the function $t \longrightarrow E[\sup_{0\leq s\leq t\wedge T_k} |X^n_s|^2]$ to deduce $E[\sup_{0\leq s\leq T\wedge T_k} |X^n_s|^2] \leq M$, where M is a constant depending only on x and T and independent from n and k. We apply Fatou's lemma by passing to the limit as k tends to $+\infty$, to deduce that $E[\sup_{0\leq s\leq T} |X^n_s|^2] \leq M$.

2) Let $0 \leq s < t < T$, we have

$$\begin{aligned} E[|X^n_t - X^n_s|^2] &= E\left[\left|\left(\int_s^t \sigma(r, X^n_{r-\frac{1}{n}}, \mathbb{P}_{X^n_{r-\frac{1}{n}}})dB_r\right) + \left(\int_s^t b(r, X^n_{r-\frac{1}{n}}, \mathbb{P}_{X^n_{r-\frac{1}{n}}})dr\right)\right|^2\right] \\ &\leq 2E\left|\int_s^t \sigma(r, X^n_{r-\frac{1}{n}}, \mathbb{P}_{X^n_{r-\frac{1}{n}}})dB_r\right|^2 + 2E\left|\int_s^t b(r, X^n_{r-\frac{1}{n}}, \mathbb{P}_{X^n_{r-\frac{1}{n}}})dr\right|^2. \end{aligned}$$

Applying Doob's and Schwartz's inequalites, we obtain

$$E[|X_t^n - X_s^n|^2] \leq 8\int_s^t E\left|\sigma(r, X^n_{r-\frac{1}{n}}, \mathbb{P}_{X^n_{r-\frac{1}{n}}})\right|^2 dr + 2(t-s)\int_s^t \left|b(r, X^n_{r-\frac{1}{n}}, \mathbb{P}_{X^n_{r-\frac{1}{n}}})\right|^2 dr.$$

By using the same arguments as in 1), we get

$$E[|X_t^n - X_s^n|^2] \leq \left(48C^2 + 12C^2(t-s)\right)\int_s^t \left[1 + E[\sup_{0\leq s\leq T}|X_s^n|^2]\right] dr.$$

Then according to property 1) there exists a positive constant N which depends only on T and C such that

$$E[|X_t^n - X_s^n|^2] \leq N(t-s).$$

Moreover if $-(\frac{1}{n}) < t,\ s \leq 0$ then by definition we have $E[|X_t^n - X_s^n|^2] = 0$. Hence for any $-(\frac{1}{n}) < t,\ s \leq T$ we have $E[|X_t^n - X_s^n|^2] \leq N(t-s)$ and property 2) is proved.$\boxdot$

The main result of this section is given by the following.

Theorem 3.2 *Let σ and b be functions satisfying conditions $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ and let (X^n) be the sequence defined by the Carathéodory scheme (3.2). Then the sequence (X^n) converges uniformly in probability to the unique solution X of equation (4.1).*

Proof. To prove the convergence of the sequence (X^n) uniformly in probability, it is enough to show that there exists a process X such that for every $T > 0$:

$$\lim_{n\to\infty} E\left[\sup_{0\leq t\leq T}|X_t^n - X_t|^2\right] = 0.$$

Let $m > n$, then

$$\begin{aligned} E\left[\sup_{0\leq t\leq T}|X_t^m - X_t^n|^2\right] &\leq 2E\left[\sup_{0\leq t\leq T}\left|\int_0^t \left(\sigma(r, X^m_{r-\frac{1}{m}}, \mathbb{P}_{X^m_{r-\frac{1}{m}}}) - \sigma(r, X^n_{r-\frac{1}{n}}, \mathbb{P}_{X^n_{r-\frac{1}{n}}})\right) dB_r\right|^2\right] \\ &\quad + 2E\left[\sup_{0\leq t\leq T}\left|\int_0^t \left(b(r, X^m_{r-\frac{1}{m}}, \mathbb{P}_{X^m_{r-\frac{1}{m}}}) - b(r, X^n_{r-\frac{1}{n}}, \mathbb{P}_{X^n_{r-\frac{1}{n}}})\right) dr\right|^2\right]. \end{aligned}$$

Using Doob's and Schwartz's inequalities we get

$$
\begin{aligned}
E\left[\sup_{0\le t\le T}|X_t^m-X_t^n|^2\right] &\le 8E\left[\int_0^T\left|\sigma(s,X^m_{s-\frac{1}{m}},\mathbb{P}_{X^m_{s-\frac{1}{m}}})-\sigma(s,X^n_{s-\frac{1}{n}},\mathbb{P}_{X^n_{s-\frac{1}{n}}})\right|^2ds\right]\\
&\quad+2TE\left[\int_0^T\left|b(s,X^m_{s-\frac{1}{m}},\mathbb{P}_{X^m_{s-\frac{1}{m}}})-b(s,X^n_{s-\frac{1}{n}},\mathbb{P}_{X^n_{s-\frac{1}{n}}})\right|^2ds\right].
\end{aligned}
$$

Using assumption $(\mathbf{H}_2)$ we get

$$
E[\sup_{0\le t\le T}|X_t^m-X_t^n|^2]\le\left(8L^2+2TL^2\right)\int_0^T E\left[\sup_{0\le r\le s}\left|X^m_{r-\frac{1}{m}}-X^n_{r-\frac{1}{n}}\right|^2+\sup_{0\le r\le s}W_2^2(\mathbb{P}_{X^m_{r-\frac{1}{m}}},\mathbb{P}_{X^n_{r-\frac{1}{n}}})\right]ds.
$$

Since

$$
W_2^2(\mathbb{P}_{X^m_{r-\frac{1}{m}}},\mathbb{P}_{X^n_{r-\frac{1}{n}}})\le E[|X^n_{r-\frac{1}{n}}-X^m_{r-\frac{1}{m}}|^2],
$$

we obtain

$$
E[\sup_{0\le t\le T}|X_t^m-X_t^n|^2]\le\left(16L^2+4TL^2\right)\int_0^T E\left[\sup_{0\le r\le s}\left|X^m_{r-\frac{1}{m}}-X^n_{r-\frac{1}{n}}\right|^2\right]ds.
$$

Let $K=16L^2+4TL^2$, then

$$
\begin{aligned}
E[\sup_{0\le t\le T}|X_t^m-X_t^n|^2]\le 2K&\int_0^T E\left[\sup_{0\le r\le s}\left|X^m_{r-\frac{1}{m}}-X^n_{r-\frac{1}{m}}\right|^2\right]ds\\
+2K&\int_0^T E\left[\sup_{0\le r\le s}\left|X^n_{r-\frac{1}{m}}-X^n_{r-\frac{1}{n}}\right|^2\right]ds.
\end{aligned}
$$

According to Lemma 3.1 , property 2) applied to the second expression in the right hand side we get

$$
E[\sup_{0\le t\le T}|X_t^m-X_t^n|^2]\le 2K\int_0^T E\left[\sup_{0\le r\le s}|X_r^m-X_r^n|^2\right]ds+2KN\left(\frac{1}{n}-\frac{1}{m}\right)T
$$

Hence, by Gronwall's lemma we get

$$
E[\sup_{0\le t\le T}|X_t^m-X_t^n|^2]\le 2KN\left(\frac{1}{n}-\frac{1}{m}\right)Te^{2KT}.
$$

If we set $A=2KNTe^{2KT}$, it is clear that A depends only on T.Therefore there exists a positive constant depending only on T such that

$$E[\sup_{0\leq t\leq T} |X_t^m - X_t^n|^2] \leq A\left(\tfrac{1}{n} - \tfrac{1}{m}\right).$$

This implies in particular that (X^n) is a Cauchy sequence in the space of square integrable continuous processes, which is a complete metric space.Therefore (X^n) converges uniformly $(P-a.s.)$ and in quadratic mean to some continuous square integrable process X .

Let $m \to \infty$ in the last inequality gives us

$$E[\sup_{0\leq t\leq T} |X_t^n - X_t|^2] \leq \frac{A}{n}.$$

Then the Borel–Cantelli Lemma can be used to show that (X^n) converges to X almost surely uniformly on $[0, T]$. Therefore $t \longrightarrow X_t$ is continuous, as a limit with respect to the uniform convergence of continuous processes. Using the same arguments as in subsection 2.2, it is easy to see that $t \longrightarrow \mathbb{P}_{X_t}$ is continuous.

Let us check that the limit X is the unique solution of the MVSDE (4.1). It is sufficient to verify that X satisfies the MVSDE (4.1), that is :

$$X_t = x + \int_0^t b(s, X_s, \mathbb{P}_{X_s})ds + \int_0^t \sigma(s, X_s, \mathbb{P}_{X_s})dB_s.$$

Let $0 < t < T$ and consider

$$\begin{aligned}
&E\left|\int_0^t \left(\sigma(r, X^n_{r-\frac{1}{n}}, \mathbb{P}_{X^n_{r-\frac{1}{n}}}) - \sigma(r, X_r, \mathbb{P}_{X_r})\right) dB_r + \int_0^t \left(b(r, X^n_{r-\frac{1}{n}}, \mathbb{P}_{X^n_{r-\frac{1}{n}}}) - b(r, X_r, \mathbb{P}_{X_r})\right) dr\right|^2 \\
&\leq \left(8L^2 + 2TL^2\right) E\int_0^t \left(\left|X^n_{r-\frac{1}{n}} - X_r\right|^2 + W_2^2(\mathbb{P}_{X^n_{r-\frac{1}{n}}}, \mathbb{P}_{X_r})\right) dr \\
&\leq \left(16L^2 + 4TL^2\right) \int_0^t E\left[\left|X^n_{r-\frac{1}{n}} - X_r\right|^2\right] dr \\
&\leq \left(16L^2 + 4TL^2\right) \left(\int_0^t E\left[\left|X^n_{r-\frac{1}{n}} - X^n_r\right|^2\right] dr + \int_0^t E\left[|X^n_r - X_r|^2\right] dr\right) \\
&\leq \frac{B}{n},
\end{aligned}$$

where $B = 32TL^2 + 8T^2L^2$ is a positive constant independent of n. We conclude by using the uniqueness of the limit and the uniqueness of the solution of our MVSDE.$\boxdot$

Remark 3.2 *The last theorem offers us an alternative to prove the existence and uniqueness* *of a strong solution. Note that in the classical proof, we use the fixed point theorem or the Picard successive approximation scheme (see [20, 45]).*

3.4 The Carathéodory scheme for non Lipschitz MVSDEs

In this section we drop the Lipschitz condition $(\mathbf{H_2})$. Instead, we assume that equation (4.1) has the pathwise uniqueness property. This means that under any condition on the coefficients ensuring pathwise uniqueness, the convergence of the Carathéodory numerical scheme still holds.

Definition 3.1 *We say that pathwise uniqueness holds for equation 4.1 if whenever X and X' are two solutions of (4.1), defined on the same probability space $(\Omega, \mathcal{F}, P)$, with common Brownian motion (B), with possibly different filtrations such that $P[X_0 = X_0'] = 1$, then X and X' are indistinguishable.*

Remark 3.3 *According to the Yamada-Watanabe theorem applied to MVSDEs [**?**], existence of a weak solution and pathwise uniqueness implies existence and uniqueness of a strong solution.*

Let us recall few results on weak convergence of stochastic processes, which play a key role in the proof of the main result.

Theorem 3.3 (*Kolmogorov criterion for tightness [44] page 18*) *Let (X_t^n), $n = 1, 2, \ldots$, be a sequence of d-dimensional continuous processes satisfying the following two conditions :*
(i) There exist positive constants M and γ such that $E[|X_0^n|^\gamma] \leq M$, for every $n = 1, 2, \ldots$..
(ii) There exist positive constants α, β, M_k, $k = 1, 2, \ldots$, such that :
$E[|X_t^n - X_s^n|^\alpha] \leq M_k |t-s|^{1+\beta}$ *for every n and $t, s \in [0, k]$, $(k = 1, 2, \ldots)$.*
Then the family of laws of (X_n) is tight.

Theorem 3.4 *(Skorokhod embedding theorem, [44] page 9) Let (S, ρ) be a complete separable metric space, $P_n, n = 1, 2, \ldots$ and P be probability measures on $(S, \mathcal{B}(S))$ such that $P_n \underset{n \to +\infty}{\longrightarrow}$*

P. Then, on a probability space $\left(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{P}\right)$, we can construct S-valued random variables X_n, $n=1,2,...,$ and X such that :

(i) $P_n=\widehat{P}^{X_n}, n=1,2,...,$ and $P=\widehat{P}^X$.

(ii) X_n converges to X, $\widehat{P}$ almost surely.

Our main convergence result of this section is based on the following simple but deep Lemma [37], which characterizes the convergence in probability in terms of weak convergence of couples of subsequences of random elements.

Lemma 3.2 *Let (E,d) be a Polish space and (X^n) a sequence of E-valued random variables. Then the following assertions are equivalent :*

1) (X^n) converges in probability to X.

2) For every pair of subsequences $\left(X^l\right)$ and (X^m), there exists a subsequence $\nu_k:=\left(X^{m(k)},X^{l(k)}\right)$ converging weakly to a random element ν supported by the diagonal $\{(x,y)\in E\times E : x=y\}$.

Proof. Suppose that the sequence (X^n) converges in probability to some random variable X, then (X^n,X^n) converges in probability to (X,X), hence weakly to the distribution of (X,X) which is supported by the diagonal.

Conversly, suppose that for every subsequences $\left(X^l\right)$ and (X^m), there exists a subsequence $\nu_k:=\left(X^{m(k)},X^{l(k)}\right)$ converging weakly to some probability measure supported by the diagonal. Then for every bounded continuous function f, $f(\nu_k)$ converges to $f(\nu)$, where ν is supported by the diagonal. Therefore if we take $f(x,y)=d(x,y)$, the metric of E, then $f(\nu_k)$ converges to $f(\nu)=0$ weakly, hence $f(\nu_k)$ converges to 0 in probability. This implies in particular that (X^n) is a Cauchy sequence on the space of random variables, equipped with metric of convergence in probability, which is complete. Then the sequence (X^n) converges in probability to some X.$\boxdot$

The main result of this section is the following theorem.

Theorem 3.5 *Assume that σ and b satisfy assumptions $(\mathbf{H_1})$. Suppose moreover that pathwise uniqueness holds for 4.1). Then the sequence (X^n) defined by the Carathéodory scheme, converges in probability to the unique solution X of equation (4.1).*

Proof. Let $T > 0$ and $t \in [0,T]$, X^n is defined by

$$X_t^n = x + \int_0^t b(s, X^n_{s-\frac{1}{n}}, \mathbb{P}_{X^n_{s-\frac{1}{n}}})dt + \int_0^t \sigma(s, X^n_{s-\frac{1}{n}}, \mathbb{P}_{X^n_{s-\frac{1}{n}}})dB_s.$$

Using the same techniques as in the proof of Lemma 3.1, we are able to show that the sequence (X^n) are continuous processes satisfying :

a) $E[\sup_{0\le s\le T} |X^n(s)|] < +\infty$,

b) $E[|X^n(t) - X^n(s)|^4] \le N\,|t-s|^2$.

Then according to Kolmogorov's tightness criterion (Theorem 4.3), (X^n) is tight as a sequence of random variables with values on the space $C\left([0,T]\,;\mathbb{R}^d\right)$ of continuous functions.

Let us consider two subsequences (X^m) and $\left(X^l\right)$ of (X^n). It is clear that (X^m) and $\left(X^l\right)$ are also tight. Moreover it is not difficult to verify that the Brownian motion (B) satisfies Kolmogorov's tightness criterion (a) and b)). Therefore (X^m, X^l, B) is tight as a sequence of $C\left([0,T]\,;\mathbb{R}^{3d}\right)$-random variables. Then according to Prokhorov's theorem the sequence of distributions of (X^m, X^l, B) is relatively compact for the topology of weak convergence of probability measures.

By Skorokhod's embedding theorem (Theorem 4.4), there exist subsequences $m(j), l(j)$ and a probability space $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}\right)$ carrying a sequence of stochastic processes $\left(\widehat{X}^{m(j)}, \widehat{Y}^{l(j)}, \widehat{B^j}\right)$ such that :

α) the distributions of $\left(X^{m(j)}, X^{l(j)}, B\right)$ and $\left(\widehat{X}^{m(j)}, \widehat{Y}^{l(j)}, \widehat{B}^j\right)$ coincide for every $j \in \mathbb{N}$, as $C\left([0,T]\,;\mathbb{R}^{3d}\right)$-random variables.

β) there exists a subsequence still denoted by $\left(\widehat{X_t}^{m(j)}, \widehat{Y_t}^{l(j)}, \widehat{B_t^j}\right)$ which converges uniformly to $\left(\widehat{X_t}, \widehat{Y_t}, \widehat{B_t}\right)$ on every finite time interval, $\widehat{P} - a.s.$

We claim that if we define the filtrations $\widehat{\mathcal{F}}_t^j = \sigma\left(\widehat{X}_s^{m(j)}, \widehat{Y}_s^{l(j)}, \widehat{B}_s^j; s \le t\right)$ and $\widehat{\mathcal{F}}_t = \sigma\left(\widehat{X}_s, \widehat{Y}_s, \widehat{B}_s; s \le t\right)$, then $\left(\widehat{B}_t^j, \widehat{\mathcal{F}}_t^j\right)$ and $\left(\widehat{B}_t, \widehat{\mathcal{F}}_t\right)$ are Brownian motions.

Indeed, it is well known that for each $j = 1, 2, ...$, the $\sigma-$field $\widehat{\mathcal{F}}_t^j$ is generated by random variables of the form $\left(\widehat{X}_{t_1}^{m(j)}, ..., \widehat{X}_{t_l}^{m(j)}, \widehat{Y}_{t_1}^{l(j)}, ..., \widehat{Y}_{t_1}^{l(j)}, \widehat{B}_{t_1}^j, ..., \widehat{B}_{t_l}^j\right)$, $0 \le t_1 \le t_2 \le ... \le t_l \le t$.

A similar claim can be made for the filtration $\widehat{\mathcal{F}}_t$.

Let g be a bounded measurable function on $\mathbb{R}^{3d}$, $\widehat{Z}_{t_i} = \left(\widehat{X}_{t_i}^{m(j)}, \widehat{X}_{t_i}^{l(j)}, \widehat{B}_{t_i}^j\right)$ and $Z_{t_i} = \left(X_{t_i}^{m(j)}, X_{t_i}^{l(j)}, B_{t_i}^j\right)$, then in view of property α)

$\widehat{E}\left[g(\widehat{Z}_{t_i})(\widehat{B}^j_t - \widehat{B}^j_s)\right] = E\left[g(Z_{t_i})(B_t - B_s)\right] = 0$, because (B_t) is a Brownian motion with respect to its own filtration and also with respect to the filtration generated by $\left(X^{m(j)}, X^{l(j)}, B\right)$. This implies that $\widehat{E}\left[\widehat{B}^j_t - \widehat{B}^j_s / \widehat{\mathcal{F}}^j_s\right] = 0$

Moreover by using the same arguments we get,

$$\begin{aligned}\widehat{E}\left[\left(\widehat{B}^j_t - \widehat{B}^j_s\right)\left(\widehat{B}^j_t - \widehat{B}^j_s\right)^t / \widehat{\mathcal{F}}^j_s\right] &= \widehat{E}\left[\left(\widehat{B}^j_t - \widehat{B}^j_s\right)\left(\widehat{B}^j_t - \widehat{B}^j_s\right)^t\right] \\ &= E\left[(B_t - B_s)(B_t - B_s)^t\right] = (t-s)I\end{aligned}$$

where X^t denotes the transpose of the vector X and I is the unit matrix.

Therefore $\left(\widehat{B^j}_t, \widehat{\mathcal{F}}^j_t\right)$ is a Brownian motion.

On the other hand using the same techniques along with property $\beta)$ on the uniform convergence of $\left(\widehat{X_t}^{m(j)}, \widehat{Y}_t^{l(j)}, \widehat{B^j_t}\right)$ to $\left(\widehat{X_t}, \widehat{Y}_t, \widehat{B_t}\right)$, we can show easily that $\left(\widehat{B}_t, \widehat{\mathcal{F}}_t\right)$ is a Brownian motion.

Now, from property $\alpha)$ and using the same techniques as in [37] Lemma 3.1, we get

$$\begin{aligned}\widehat{X}^{m(j)}_t &= x + \int_0^t \sigma\left(s, \widehat{X}^{m(j)}_{s-\frac{1}{m(j)}}, \mathbb{P}_{\widehat{X}^{m(j)}_{s-\frac{1}{m(j)}}}\right) d\widehat{B^j_s} + \int_0^t b\left(s, \widehat{X}^{m(j)}_{s-\frac{1}{m(j)}}, \mathbb{P}_{\widehat{X}^{m(j)}_{s-\frac{1}{m(j)}}}\right) ds, \\ \widehat{Y}^{l(j)}_t &= x + \int_0^t \sigma\left(s, \widehat{Y}^{l(j)}_{s-\frac{1}{l(j)}}, \mathbb{P}_{\widehat{Y}^{l(j)}_{s-\frac{1}{l(j)}}}\right) d\widehat{B^j_s} + \int_0^t b\left(s, \widehat{Y}^{l(j)}_{s-\frac{1}{l(j)}}, \mathbb{P}_{\widehat{Y}^{l(j)}_{s-\frac{1}{l(j)}}}\right) ds.\end{aligned}$$

The process $\widehat{X}^{m(j)}_s$ is adapted to its own filtration then adapted to $\left(\widehat{\mathcal{F}}^j_s\right)$, then $\sigma\left(s, \widehat{X}^{m(j)}_{s-\frac{1}{m(j)}}, \mathbb{P}_{\widehat{X}^{m(j)}_{s-\frac{1}{m(j)}}}\right)$ and $b\left(s, \widehat{X}^{m(j)}_{s-\frac{1}{m(j)}}, \mathbb{P}_{\widehat{X}^{m(j)}_{s-\frac{1}{m(j)}}}\right)$ are also $\widehat{\mathcal{F}}^j_s-$adapted because the mappings b and σ satisfy are measurable in t and continuous in the (x, μ).

According to property $\beta)$ and using Skorokhod's limit theorem ([65] or [37], Lemma 3.1) it holds that,

$$\int_0^t \sigma\left(s, \widehat{X}^{m(j)}_{s-\frac{1}{m(j)}}, \mathbb{P}_{\widehat{X}^{m(j)}_{s-\frac{1}{m(j)}}}\right) d\widehat{B^j_s} \xrightarrow[j\to\infty]{\widehat{P}} \int_0^t \sigma\left(s, \widehat{X}_s, \mathbb{P}_{\widehat{X}_s}\right) d\widehat{B}_s,$$

$$\int_0^t b\left(s, \widehat{X}^{m(j)}_{s-\frac{1}{m(j)}}, \mathbb{P}_{\widehat{X}^{m(j)}_{s-\frac{1}{m(j)}}}\right) ds \xrightarrow[j\to+\infty]{\widehat{P}} \int_0^t b\left(s, \widehat{X}_s, \mathbb{P}_{\widehat{X}_s}\right) ds.$$

uniformly in probability.

We conclude, as $j \to \infty$, that $\widehat{X}$ and $\widehat{Y}$ satisfy the same MVSDE (4.1) on the new probability space $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}\right)$, with the same initial condition x and common Brownian motion $\widehat{B}$,

$$\widehat{X_t} = x + \int_0^t \sigma\left(s, \widehat{X_s}, \mathbb{P}_{\widehat{X_s}}\right) d\widehat{B_s} + \int_0^t b\left(s, \widehat{X_s}, \mathbb{P}_{\widehat{X_s}}\right) ds,$$

and

$$\widehat{Y}_t = x + \int_0^t \sigma\left(s, \widehat{Y}_s, \mathbb{P}_{\widehat{Y}_s}\right) d\widehat{B_s} + \int_0^t b\left(s, \widehat{Y}_s, \mathbb{P}_{\widehat{Y}_s}\right) ds.$$

According to the pathwise uniqueness for (4.1) it holds that $\widehat{X} = \widehat{Y}$, $\widehat{P}-$a.s. Therefore there exists a subsequence of $\left(\widehat{X}_t^{m(j)}, \widehat{X}_s^{l(j)}\right)$ which converges uniformly to $\left(\widehat{X}, \widehat{X}\right)$. This means that $\left(X^{m(j)}, X^{l(j)}\right)$ converges weakly to a probability distribution supported by the diagonal. Hence, according to Lemma 4.5, the sequence (X^n) converges in probability to the unique solution of MVSDE (4.1).$\boxdot$

Using the same arguments as in Theorem 4.6., we are able to prove the following corollary.

Corollary 3.1 *Assume that σ and b satisfy assumption $(\mathbf{H_1})$and that the uniqueness in law holds for MVSDE (4.1). Then the sequence (X^n) of delayed processes (3.2) converges weakly to the unique solution X of (4.1).*

3.4.1 Examples

It is clear that Theorem 4.6 extends the Lipschitz case, and is valid under any set of assumptions ensuring the pathwise uniqueness of the solution. In what follows, we give concrete examples of MVSDEs, with non Lipshitz coefficients, under which the Carathéodory numerical scheme converges strongly to the unique solution, while there is no evidence of the convergence of the Picard numerical scheme.

Corollary 3.2 *Assume that the coefficients b and σ are time independent and satisfy the following conditions.*

($\boldsymbol{H}_1$)The drift coefficient $b : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d$ is of the form $b(x, \mu) = -\nabla U(x) + v(x, \mu)$ where U is convex and of class C^1. The function b is assumed to be globally Lipschitz in both variables, and for all $x \in \mathbb{R}^d$, we have $\sup_{\mu\in\mathcal{P}_2(\mathbb{R}^d)} |b(x, \mu)| < +\infty$.

($\boldsymbol{H}_2$) The diffusion coefficient $\sigma : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d$ satisfies the usual global Lipschitz condition in both variables. Moreover, for all $x \in \mathbb{R}^d$, we have $\sup_{\mu\in\mathcal{P}_2(\mathbb{R}^d)} |\sigma(x, \mu)| < +\infty$.

Then the Carathéodory scheme converges strongly.

Proof. According to [4], Theorem 4.1, the MVSDE (4.1) has a unique strong solution under ($\mathbf{H_1}$-$\mathbf{H}_2$), then the result follows by Theorem 4.6.⊡

Corollary 3.3 *Assume that the coefficients are one dimensional, σ does not depend on the probability measure μ and the coefficients of the MVSDE(4.1) satisfy :*

($\mathbf{A_1}$) *There exist $C > 0$, such that : $|b(t, x, \mu) - b(t, x, \nu)| \leq CW_2(\mu, \nu)$*

($\mathbf{A_2}$) *There exist $K > 0$, such that for every $x, y \in \mathbb{R}$, $|\sigma(t, x) - \sigma(t, y)| \leq K|x - y|)$.*

($\mathbf{A_3}$) *(Osgood type condition) There exists a strictly increasing function $\kappa(u)$ on $[0, +\infty)$ such that $\kappa(0) = 0$ and κ is concave satisfying $\int\limits_{0^+} \kappa^{-1}(u)du = +\infty$, such that for every $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ and $\mu \in \mathcal{P}_1(\mathbb{R})$, $|b(t, x, \mu) - b(t, y, \mu)| \leq \kappa(|x - y|)$.*

Then the Carathéodory scheme converges strongly.

Proof. According to [8] the MVSDE (4.1) has a unique strong solution under ($\mathbf{A_1}$-$\mathbf{A}_3$), then the result follows by Theorem 4.6.⊡

Corollary 3.4 *Assume that the coefficients of MVSDE (4.1) satisfy the following conditions :*

1) σ is Lipschitz in (x, μ) uniformly in t. There exist $L > 0$ such that. for any $x,\ y \in \mathbb{R}^d$ and $\mu,\ \nu \in \mathcal{P}_2(\mathbb{R}^d)$:

$|\sigma(t, x, \mu) - \sigma(t, y, \nu)| \leq L(|x - y| + W_2(\mu, \nu)$.

2) One-sided Lipschitz in x and Lipschitz in law :there exist constants $C, K > 0$, such that. for any $t \in [0, T]$, $x,\ y \in \mathbb{R}^d$ and $\mu,\ \nu \in \mathcal{P}_2(\mathbb{R}^d)$:

$\langle x - y, b(t, x, \mu) - b(t, y, \mu)\rangle \leq C. |x - y|^2$

$|b(t,x,\mu) - b(t,x,\nu)| \leq K.W_2(\mu,\nu)$.

3) Locally Lipschitz with polynomial growth in x : there exist $L > 0$, $q \in N$ with $q > 1$, such that for all $t \in [0,T]$, $\forall \mu \in \mathcal{P}_2(\mathbb{R}^d)$ and all $x, y \in \mathbb{R}^d$

$|b(t,x,\mu) - b(t,y,\mu)| \leq L\left(1 + |x|^q + |y|^q\right)|x - y|$

4) b and σ are 1/2-Hölder continous in time.

Then the Carathéodory scheme converges strongly to the unique solution.

Proof. According to [28], the MVSDE (4.1) has a unique strong solution under the above assumptions, then the result follows by Theorem 4.6.⊡

Corollary 3.5 *Let us assume that the MVSDE(4.1) has the form :*

$$X_t = x + \int_0^t b(s, X_s, E\left[\varphi_1(X_s)\right])ds + \int_0^t \sigma(s, X_s, E\left[\varphi_2(X_s)\right])dB_s,$$

satisfying the following conditions :

$(\mathbf{B}_1)$ *b is measurable, uniformly bounded, Hölder continuous in the second variable uniformly in the first and third variables. $b(t,x,.)$ is differentiable with bounded derivative and φ_1 is Hölder continuous with exponent $0 < \alpha \leq 1$.*

$(\mathbf{B}_2)$ *$\sigma(t,.,.)$ is globally Lipschitz in the second and third variables uniformly in the first. $\sigma(t,x,.)$ is differentiable in the third variable and its derivative $\sigma_y(t,x,.)$is bounded. Moreover the derivative $\sigma_y(t,x,.)$ is Hölder continuous with respect to the second variable x.*

$(\mathbf{B}_3)$ *There exists $\lambda > 1$ such that for every $\xi \in \mathbb{R}^d$: $\frac{1}{\lambda} \leq |\langle \sigma\sigma^*\xi, \xi\rangle| \leq \lambda\left|\xi\right|^2$*

Then the Carathéodory scheme converges strongly to the unique solution.

Proof According to [21] Theorem 1.1, the MVSDE (4.1) has a unique strong solution under $\mathbf{B_1} - \mathbf{B}_3$. Then the result follows by Theorem 4.6.⊡

Remark 3.4 *It is well known, even for ordinary differential equations, that existence and uniqueness are not sufficient for the convergence of the Picard successive approximation scheme. Therefore Theorem 4.1 offers an alternative to the Picard scheme. Indeed, unlike the Picard scheme, the Carathéodory scheme converges under any conditions ensuring the pathwise uniqueness.*

3.5 Conclusion

We have considered approximation of Mckean-Vlasov stochastic differential equations, by using Carathéodory numerical scheme. This is obtained by introducing small delays in the original equation. The delayed processes are computed recursively, by successive stochastic integration on small time intervals. We proved that, under Lipschitz type hypothesis on the coefficients that $E[\sup_{0\leq t\leq T} |X^n(t) - X(t)|^2] \leq \frac{B}{n}$. This means that the sequence of delayed processes converges strongly to the unique strong solution of the MVSDE, with a rate of convergence of order $1/\sqrt{n}$. In our second main result, we have proved that the result remains valid, provided the MVSDE has the property of pathwise uniqueness. There are two main advantages of the Carathéodory numerical procedure, if we compare it with the classical Picard approximation scheme. The first advantage is that when computing X^k, there is no need to compute the preceding approximate solutions $X^1 X^2, ...X^{k-1}$ as in the Picard scheme. The other advantage is that the approximate solutions converge strongly to the unique solution under any conditions on the coefficients ensuring the pathwise uniqueness. As it is well known, this property is no more valid for the convergence of the Picard successive approximations. Indeed, even in deterministic ordinary differential equations, the uniqueness is not sufficient to ensure the convergence of the Picard scheme, see [**?**] and the references therein. Examples are given of MVSDEs with unique solutions and non Lipschitz coefficients, for which the Carathéodory scheme converges, while there is no evidence for the convergence of the Picard scheme.

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Chapitre 4

Compactification in optimal control of McKean-Vlasov stochastic differential equations

ABSTRACT. We study existence and approximation of optimal controls for systems governed by McKean-Vlasov stochastic differential equations. It is well known in simple examples that the usual control problem where admissible controls are finite dimensional valued processes has no optimal solution. The compactification of the set of such strict controls leads to measure valued controls called relaxed controls. We prove that under pathwise uniqueness of solutions of the driving equation, the relaxed controlled trajectory may be strongly approximated by a sequence of strictly controlled trajectories. This means that the relaxed and strict control problems have the same value function. Moreover, we show, under merely continuity of the coefficients, that an optimal control exists in the space of relaxed controls. Under additional convexity hypothesis, we show that the optimal relaxed control is in fact a strict control. These two results extend known results to general non linear MVSDEs and under minimal assumptions on the coefficients.

Key words : McKean-Vlasov stochastic differential equation – Martingale measure – Pathwise uniqueness – Existence – Approximation – Relaxed control - .

2010 Mathematics Subject Classification. 60H10, 60H07, 49N90.

4.1 Introduction

In this paper, we consider optimal control problems, where the system is governed by a McKean-Vlasov stochastic differential equations (MVSDE), called also mean-field stochastic differential equation (MFSDE) :

$$\begin{cases} dX_t = b\left(t, X_t, \mathbb{P}_{X_t}, u_t\right) dt + \sigma\left(t, X_t, \mathbb{P}_{X_t}, u_t\right) dB_t \\ X_0 = x \end{cases} \tag{4.1}$$

The expected cost on the time interval $[0, T]$ has the form :

$$J(u) = E\left[\int_0^T h(t, X_t, \mathbb{P}_{X_t}, u_t) dt + g(X_T, \mathbb{P}_{X_T})\right]. \tag{4.2}$$

The objective is to minimize the cost functional $J(u)$ on the space $\mathcal{U}_{ad}$, that is find u^* such that $J(u^*) = \min\left\{J(u), u \in \mathcal{U}_{ad}\right\}$.

McKean-Vlasov stochastic differential equations have appeared for the first time in a seminal paper by H. P. McKean [56] in his probabilistic study of Vlasov partial differential equation [46, **?**], modelling the evolution of large systems of particles in statistical physics. An excellent account is given in Snitzman lecture notes [66]. These equations were obtained as limits or average of some weakly interacting particle systems as the number of particles tends to infinity. One can refer to [20, 45, 66] for details on the existence and uniqueness of solutions for such SDEs. Recently McKean-Vlasov stochastic differential equations appeared to be the natural setting of mean-field games (MFG) theory, introduced independently by P.L. Lions and J.M. Lasry [51] and Huang, Malhamé Caines [43] in 2006. The objective of MFG theory is to solve the problem of the existence of an approximate Nash equilibrium for differential games, with a large number of small players. Since the earlier papers, a huge literature has been developped on MFG theory, motivated by applications to various real life problems such as game theory, mathematical finance, communications networks and study of large populations. We refer to [14] for an early complete survey on the subject.

Motivated by differential games, control problems where the state process is a MVSDE, where the coefficients depend on the marginal probability law of the solution, have been studied in [20, 51] and provide interesting models in applications, in particular to mathematical finance problems. A typical example is the continuous-time Markowitz's mean-variance portfolio selection problem, where one should minimize an objective function involving a quadratic function of the expectation, due to the variance term. Due to inconsistency problems the dynamic programming in its original form does not hold. Therefore the initial efforts were devoted to establishing optimality necessary conditions of the Pontriagin type (see [3, 19]). Recently a Hamilton- Jacobi Bellmann equation on the Wasserstein space of probability measures has been derived using differentiability with respect to probability measures (see [63]). One can refer to [20] as the most recent and updated reference on mean-field games and mean-field control and the complete list of references therein.

Our main goal in this paper is to study existence, as well as some approximation properties of optimal controls. Our starting point is a control problem (called strict control problem), which does not necessarily admit an optimal control. We construct a second control problem, called the relaxed control problem, with two main properties. The first one is that the relaxed problem admits an optimal solution. The second property is that the relaxed controls as well as their states and cost functionals could be approximated by means of strict controls and their states and cost functionals. This last stability property is useful in numerical and engineering problems. Indeed, it is more convenient to handle nearly optimal controls which are functions instead of optimal controls which are measure valued processes. To achieve this objective we apply the so-called compactification method to show the existence of an optimal relaxed control. This method is based on the relative compactness of the distributions of the processes under consideration, which does not require any regularity of the coefficients or the value function.

It is well known in deterministic as well as in stochastic control problems that an optimal control does not necessarily exist in the space of strict controls $\mathcal{U}_{ad}$, if we don't impose additional convexity conditions. To overcome this serious drawback, the methodology is to use generalized curves introduced long time ago by Young [70] in the framework of the calculus of variations and known as *Young measures*, or *relaxed controls* in deterministic optimal control. The idea is to identify an admissible strict control $u(t, \omega)$ as a positive measure on $[0, T] \times \mathbb{A}$ defined by $\mu(dt, da) = dt.\delta_{u(t)}(da)$. The closure $\mathbb{V}$ of the set of such mesures under weak convergence topology is the set of positive measures on $[0, T] \times \mathbb{A}$, whose projection on $[0, T]$ is the Lebeasgue measure. The set $\mathbb{V}$ enjoys nice compatness and convexity properties. This approach has been developped in deterministic control theory, in particular in [70] and in stochastic control of diffusion processes in [5, 30, 39]. Another general approach using occupation measures has been developped in [49], to handle general controlled martingale problems. Existence of optimal controls for a particular class of McKean-Vlasov SDEs with Lipschitz coefficients, where the dependence upon the marginal probability measure is made via a linear functional, has been treated in [6, 7].

We prove two main results. The first is a strong approximation result of the relaxed control problem by a sequence of strict control problems. This means in particular that relaxing out the initial problem does not affect the value function of our problem. This means that the strict and relaxed control problems have in fact the same value function. This is performed under merely continuous coefficients and pathwise uniqueness of the solutions of the state equation. This result could be seen as a stability result with respect to the control variable and could be used to perform numerical approximations. The second main result is the existence of an optimal relaxed control under continuity of the coefficients. Note that our results improve known existence of optimal controls for systems driven by Itô SDEs [5, **?**] as well as McKean-Vlasov SDEs [6, 7] in two directions. The first one is that we consider general non linear McKean-Vlasov SDEs and not only those where the coefficients depend upon some linear functional of the marginal probability measure. The second novelty is that our results are proved under minimal assumptions of the coefficients, namely the continuity and boundedness of the coefficients. The continuity ensures that the limiting procedures make sense and the boundedness ensures the relative compactness of the distributions of the processes under consideration. The main ingredients used in the proofs are the tightness criteria and Skorokhod selection theorem.

4.2 Formulation of the problem

Let (B_t) is a $d-$dimensional Brownian motion, defined on some probability space $(\Omega, \mathcal{F}, P)$, equipped with a filtration $(\mathcal{F}_t)$, satisfying the usual conditions. Let $\mathbb{A}$ be some compact metric space called the action space.

We denote $\mathcal{P}_2(\mathbb{R}^d)$ the space of probability measures on $\mathbb{R}^d$, with finite second order moment, equipped with the topology induced by the 2-Wasserstein metric. For $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, define the 2-Wasserstein distance $W_2(\mu, \nu)$ by :

$$W_2(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} [\int_{E\times E} |x-y|^2 \, d\pi(x,y)]^{1/2}$$

where $\Pi(\mu, \nu)$ denotes the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ whose first and second marginals are respectively μ and ν. In the case $\mu = \mathbb{P}_X$ and $\nu = \mathbb{P}_Y$ are the laws of $\mathbb{R}^d$-valued random variable X and Y having second order moments, then

$$W_2(\mu, \nu)^2 \leq \mathbb{E}[\,|X-Y|^2].$$

Let us assume the following conditions :

$(\mathbf{H_1})$ Assume that

$$\begin{aligned}
&b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{A} \longrightarrow \mathbb{R}^d \\
&\sigma : [0,T] \times \mathbb{R}^d \times \mathcal{P}\left({}_2\mathbb{R}^d\right) \times \mathbb{A} \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d \\
&h : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{A} \longrightarrow \mathbb{R}^d \\
&g : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d
\end{aligned} \tag{4.3}$$

are bounded and continuous functions.

Consider the following control problem

$$\begin{cases} \text{Mimimize } J(u) = E\left[\int_0^T h(t, X_t, \mathbb{P}_{X_t}, u_t)dt + g(X_T, \mathbb{P}_{X_T})\right] \\ \text{subject to } X_t = x + \int_0^t b\left(s, X_s, \mathbb{P}_{X_s}, u_s\right) ds + \int_0^t \sigma\left(s, X_s, \mathbb{P}_{X_s}, u_s\right) dB_s \end{cases}$$

over the set $\mathcal{U}_{ad}$ of admissible controls (called strict controls) which are progressively measurable processes with values on the space of actions $\mathbb{A}$.

By analogy to the usual Itô SDEs, let L^ν be the infinitesimal generator associated with the solution of equation (4.1),

$$L^\nu f(t,x,a) = \frac{1}{2}\sigma\sigma^* \frac{\partial^2 f}{\partial x^2}(t,x,a) + b\frac{\partial f}{\partial x}(t,x,a),$$

where $b = b(t,x,\nu,a)$ and $\sigma\sigma^* = \sigma\sigma^*(t,x,\nu,a)$ for $\nu \in \mathcal{P}_2(\mathbb{R})$.

Definition 4.1 *A strict control is the term $\alpha = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P, u, X, x)$ such that*

(1) $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions.

(2) u is an $\mathbb{A}$-valued process, progressively measurable with respect to $(\mathcal{F}_t)$.

(3) (X_t) is $\mathbb{R}^d$-valued, $\mathcal{F}_t$-adapted, with continuous paths, such that

$$f(X_t) - f(x) - \int_0^t L^{\mathbb{P}_{X_s}} f(s, X_s, u_s)\, ds \quad \textit{is a } P-\textit{martingale}, \tag{4.4}$$

for every $f \in C_b^2$.

We denote by $\mathcal{U}_{ad}$ the space of strict controls. It is proved in [45, 52] that the existence of a solution of the martingale problem (4.4) is equivalent to the existence of a weak solution of (4.1).

The controls as defined in the last definition are called weak controls, because of the possible change of the probability space and the Brownian motion with the control u_t.

Remark 4.1 *a) Statement (3) in the last definition is equivalent to say that B is a Brownian motion and (B, X) is a solution of the MVSDE (4.1)*

b) It is well known that if the coefficients are globally Lipschitz in $(x,\mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, then the MVSDE (4.1) has a unique strong solution for each fixed strict control.

c) Under continuity and boundedness of the coefficient, for each fixed constant strict control, (4.1) admits a weak solution [45, 52] . This means in particular that the set $\mathcal{U}_{ad}$ of strict controls is not empty. If we suppose in addition that the pathwise uniqueness property holds *then the Yamada-Watanabe theorem holds, that is the MVSDE (4.1), has a strong solution (see* [**?**]*).*

Example : The mean-variance problem.

Consider a market, in which two securities with prices S^0 and S^1 where :

$$\begin{cases} dS_t^0 = S_t^0 \rho_t dt,\ S_0^0 = s_0 > 0 \\ dS_t^1 = S_t^1 \left(b_t dt + \sigma_t dB_t\right),\ S_0^1 = s^1 > 0 \end{cases}$$

S_t^0 at time $t \in [0,T]$ is a bond and S_t^1 is a stock and $\rho_t > 0, b_t, \sigma_t$ are deterministic and bounded functions.

A portfolio π is a process representing the amount of money invested in the stock. The wealth process $x^{x_0,\pi}$ corresponding to initial capital $x_0 > 0$, and portfolio π, satisfies then the equation

$$\begin{cases} dx_t = \left(\rho_t x_t + \pi_t \left(b_t - \rho_t\right)\right) dt + \pi_t \sigma_t dB_t,\ \text{for } t \in [0,T], \\ x_0 = x. \end{cases}$$

The mean-variance problem is to maximize the mean terminal wealth $\mathbb{E}\left[x_T^\pi\right]$, and to minimize the variance of the terminal wealth $Var\left[x_T^\pi\right]$, over controls π valued in $\mathbb{R}$. Then, the mean-variance portfolio optimization problem is to minimize the cost J, given by

$$J(\pi) = -\mathbb{E}\left[x_T\right] + \mu Var\left[x_T\right],$$

The admissible portfolio is assumed to be progressively measurable square integrable process, and such that the corresponding $x_t^\pi \geq 0$, for all $t \in [0,T]$. We denote by Π the class of such strategies.

Note that the cost functional may be rewritten in mean-field terms as

$$J(\pi) = \mathbb{E}\left[-x_T + \mu\left(x_T + \mathbb{E}\left[-x_T\right]\right)^2\right].$$

in which the dynamics are classical Itô SDE while the cost functional is a of mean-field type in the sense that it depends on the state process as well as on its distribution via the expectation.

Let us recall some results on the tightness and relative compactness of families of probability measures.

Definition 4.2 *(Tightness) Let E be a metric space and $\mathcal{P}(E)$ the set of all probability measures on E. We say that a family S in $\mathcal{P}(E)$ is tight if for every $\varepsilon > 0$ there exist a compact $K_\varepsilon \subset E$ such that $\mu(K_\varepsilon) < \varepsilon$, for all $\mu \in S$.*

Theorem 4.1 *(Prokhorov's Theorem [44]) Let S be a subset of $\mathcal{P}(E)$.*

1) If S is tight then it is relatively compact for the topology of weak convergence..

2) If E is complete separable complete metric space and S relatively compact then S is tight.

Theorem 4.2 (*Kolmogorov criterion for tightness* [44] *page 18*) *Let $(X_n(t))$, $n = 1, 2, ...,$ be a sequence of d-dimensional continuous processes satisfying the following two conditions :*

(i) There exist positive constants M and γ such that $E\left[|X_n(0)|^\gamma\right] \leq M$ for every $n = 1, 2, ...$.

(ii) There exist positive constants α, β, M_k, $k = 1, 2, ...,$ such that :

$E\left[|X_n(t) - X_n(s)|^\alpha\right] \leq M_k |t-s|^{1+\beta}$ for every n and $t, s \in [0, k]$, $(k = 1, 2, ...)$.

Then the family (X_n) is tight in C.

Theorem 4.3 *(Skorokhod selection theorem, [44] page 9) Let (S, ρ) be a complete separable metric space, $P_n, n = 1, 2, ...$ and P be probability measures on $(S, \mathcal{B}(S))$ such that $P_n \underset{n\to+\infty}{\longrightarrow} P$. Then, on a probability space $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}\right)$, we can construct S-valued random variables X_n, $n = 1, 2, ...,$ and X such that :*

(i) $P_n = \widehat{P}^{X_n}, n = 1, 2, ...,$ and $P = \widehat{P}^X$.

(ii) X_n converges to X, $\widehat{P}$ almost surely.

The main approximation result will be proved merely under pathwise uniqueness of solutions of the state equation. Let us give the precise definition of pathwise uniqueness for equation (4.1).

Definition 4.3 *We say that pathwise uniqueness holds for equation (4.1) if X and X' are two solutions defined on the same probability space $(\Omega, \mathcal{F}, P)$ with common Brownian motion (B), with possibly different filtrations such that $P\left[X_0 = X_0'\right] = 1$, then X and X' are indistinguishable.*

4.3 The relaxed control problem

4.3.1 A typical example from deterministic control

Let us consider a rather classical control problem, where a strict optimal control does not exist, while an optimal relaxed control exists. This is a typical example considered in the calculus of variation and in deterministic control theory. Consider the control problem

$$\begin{cases} \text{Mimimize } J(u) = \int_0^1 X^u(t)^2 dt \\ \text{subject to } X_t^u = \int_0^t u_s ds \end{cases}$$

over the set $\mathcal{U}_{ad}$ of admissible controls, which are measurable functions $u : [0,1] \to \{-1, 1\}$. $\mathcal{U}_{ad}$ is simply the space $L^0\left([0,T], \mathbb{A}\right)$ of measurable functions with values in $\mathbb{A}$. When equipped with the topology of convergence in measure $L^0\left([0,T], A\right)$ is a complete separable metric space, while compact sets in $L^0\left([0,T], A\right)$ are difficult to characterize. In fact the topology of convergence in measure is too strong and one should look for a weak topology where there are a lot of compact sets.

Consider the sequence of Rademacher functions (u_n) :

$$u_n(t) = (-1)^k \text{ if } \tfrac{k}{n} \leq t < \tfrac{k+1}{n},\ 0 \leq k \leq n-1.$$

The idea of relaxed control is to replace the $\mathbb{A}-$valued process (u_t) with $\mathcal{P}_2(\mathbb{A})-$valued process (μ_t). If we identify $u_n(t)$ with the measure $\mu_n(dt, du) = dt\delta_{u_n}(t)(du)$, we get a measure on $[0,1]\times\mathbb{A}$. A relaxed control is a measure on $[0,1]\times\mathbb{A}$ whose projection on $[0,1]$ is the Lebesgue measure dt. The relaxed control problem extending the initial strict control problem is then defined as follows.

$$\begin{cases} \text{Mimimize } J(u) = \int_0^1 X^\mu(t)^2 dt \\ \text{subject to } X_t^\mu = \int_0^t \int_{\mathbb{A}} a\mu(s, da) ds \end{cases}$$

over the set $\mathcal{R}$ of relaxed controls.

Lemma 4.1 *1) If (u_n) is the Rademacher function then* $\lim_{n\longrightarrow+\infty} J(u_n) = 0$.

2) $\inf_{u\in\mathcal{U}_{ad}} J(u)=0$ but there is no optimal control on the space $\mathcal{U}_{ad}$ of strict admissible controls.

2) The sequence $(\mu_n(dt,du))_n$ converges weakly to $dt\cdot(1/2)[\delta_{-1}+\delta_1](da)$ which is an optimal relaxed control.

Proof. 1) It is easy to check that $|X^{u_n}(t)|\leq 1/n$ and $|J(u_n)|\leq 1/n^2$ which implies that $\inf_{u\in U_{ad}} J(u)=0$.

2) There is however no control $u\in\mathcal{U}_{ad}$ such that $J(u)=0$. If this would have been the case, then for every t, $X^u(t)=0$. This in turn would imply that $u_t=0$, which is not allowed as u takes its values in $\{-1,1\}$.

3) It is sufficient to show that for every bounded continuous function $f:[0,1]\times\mathbb{A}\longrightarrow\mathbb{R}$

$$\iint_{[0,1]\times\mathbb{A}} f(t,a)\mu_n(dt,du) \text{ converges to } \iint_{[0,1]\times\mathbb{A}} f(t,a)\mu(dt,du)=\frac{1}{2}\left(\int_{[0,1]} f(t,-1)dt+\int_{[0,1]} f(t,-1)dt\right),$$

as $n\longrightarrow+\infty$.

Consider the case $n=2m$.

$$\iint_{[0,1]\times\mathbb{A}} f(t,a)\mu_n(dt,du)=\sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n} f(t,(-1)^k)dt=$$
$$\sum_{k=0}^{m-1}\int_{2j/2m}^{(2j+1)/2m} f(t,1)dt+\sum_{k=0}^{m-1}\int_{(2j+1)/2m}^{(2j+2)/2m} f(t,-1)dt$$

$f(t,-1)$ and $f(t,1)$ are continuous on $[0,1]$ which is bounded and closed, then they are uniformly continuous. Then for some $\varepsilon>0$, there exists $N\in\mathbb{N}^*$ such that for every $m\geq N$ such that $|t-s|<\frac{1}{m}$ we have $|f(t,a)-f(s,a)|<\varepsilon$, for $a=1$ or $a=-1$.

This implies in particular that

$$\left|\int_{2j/2m}^{(2j+1)/2m} f(t,a)dt-\int_{(2j+1)/2m}^{(2j+2)/2m} f(t,a)dt\right|<\frac{\varepsilon}{2m}, \text{ for } j=0,1,...,m-1$$

and therefore

$$\left|\sum_{j=0}^{m-1}\int_{2j/2m}^{(2j+1)/2m} f(t,a)dt-\sum_{j=0}^{m-1}\int_{(2j+1)/2m}^{(2j+2)/2m} f(t,a)dt\right|<\frac{\varepsilon}{2}$$

But we know that

$$\sum_{j=0}^{m-1}\int_{2j/2m}^{(2j+1)/2m} f(t,a)dt+\sum_{j=0}^{m-1}\int_{(2j+1)/2m}^{(2j+2)/2m} f(t,a)dt=\int_{[0,1]} f(t,a)dt$$

Therefore

$$\lim_{m\to+\infty}\sum_{j=0}^{m-1}\int_{2j/2m}^{(2j+1)/2m} f(t,a)dt=\lim_{m\to+\infty}\sum_{j=0}^{m-1}\int_{(2j+1)/2m}^{(2j+2)/2m} f(t,a)dt=1/2\int_{[0,1]} f(t,a)dt\ ,\ a=1\text{ or }-1.$$

and

$$\begin{aligned}\lim_{n\to+\infty}\sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n} f(t,(-1)^k)dt&=\tfrac{1}{2}\left(\int_{[0,1]} f(t,1)dt+\int_{[0,1]} f(t,-1)dt\right)\\&=\int_0^1\int_{\mathbb{A}} f(t,a)\tfrac{1}{2}\left(\delta_{-1}+\delta_1\right)(da)dt\end{aligned}$$

which achieves the proof.

Let $\mu=\frac{1}{2}\left(\delta_{-1}+\delta_1\right)(da)dt$ and X^{μ} the corresponding state process, then $X_t^{\mu}=0$ and therefore $J(\mu)=0$, which implies that μ is optimal. $\boxdot$

4.3.2 Martingale measures

Martingale measures have been introduced by Walsh [69] to study stochastic partial differential equations. They will play the role of the Brownian motion in the pathwise representation of the solution of the martingale problem associated with relaxed controls.

Definition 4.4 *Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P)$ be a filtered probability space and $(E,\mathcal{E})$ a metric space $\{M_t(A),t\geqq 0,A\in\mathcal{E}\}$ is called a $\mathcal{F}_t-$martingale measure if and only if :*

1) $\{M_t(A),t\geqq 0\}$ is a $\mathcal{F}_t$-martingale, $\forall A\in\mathcal{E}$;

2) $\forall t>0, M_t(.)$ is a L^2-valued σ-finite measure in the following sense : there exists a non-decreasing sequence (E_n) of E with $\cup_n E_n=E$ such that :

a) for every $t>0$, $\sup\limits_{A\in\mathcal{E}_n} E\left[M(A,t)^2\right]<\infty$, $\mathcal{E}_n=\mathcal{B}(E_n)$

b) for every $t>0$, $E\left[M(A_j,t)^2\right]\to 0$, for all sequence A_j of $\mathcal{E}_n$ decreasing to $\varnothing$.

For $A,B\in\mathcal{E}$, there exists a unique predictable process $\langle M(A),M(B)\rangle_t$, such that

$$M(A,t)M(B,t)-\langle M(A),M(B)\rangle_t\ \text{ is a martingale.}$$

A martingale measure M is called orthogonal if $M(A,t).M(B,t)$ is a martingale for $A, B \in \mathcal{E}$, $A \cap B = \varnothing$.

If M is an orthogonal martingale measure, one can prove the existence of random σ-finite positive measure $\mu(ds, dx)$ on $\mathbb{R} \times E$, $\mathcal{F}_t-$predictable, such that for each B of $\mathcal{A}$ the process $(\mu((0,t] \times B))_t$ is predictable and satisfies

$$\forall B \in \mathcal{E}, \forall t > 0, \quad \mu((0,t] \times B) = \langle M(B) \rangle_t \quad P - a.s.$$

We refer to [69] and [29] for a complete construction of the stochastic integral with respect to orthogonal martingale measures.

4.3.3 The relaxed control problem

Let $\mathbb{V}$ be the set of product measures μ on $[0,T] \times \mathbb{A}$ whose projection on $[0,T]$ coincides with the Lebesgue measure dt. $\mathbb{V}$ as a closed subspace of the space of positive Radon measures $\mathbb{M}_+([0,T] \times \mathbb{A})$ is compact for the topology of weak convergence.

Definition 4.5 *A measure valued control on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a random variable $\mu = dt.\mu_t(da)$ with values in $\mathbb{V}$, such that $\mu_t(da)$ is progressively measurable with respect to $(\mathcal{F}_t)$ and such that for each t, $1_{(0,t]}.\mu$ is $\mathcal{F}_t-$measurable.*

Definition 4.6 *A relaxed control is the term $\alpha = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P, \mu, X, x)$ such that*

(1) $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions ;

(2) μ is measure valued control, progressively measurable with respect to $(\mathcal{F}_t)$;

(3) (X_t) is $\mathbb{R}^d$-valued, $\mathcal{F}_t$-adapted, with continuous paths, such that

$$f(X_t) - f(X_0) - \int_0^t \int_{\mathbb{A}} L^{\mathbb{P}_{X_s}} f(s, X_s, a)\, \mu_s(da) ds \text{ is a } P - martingale, \tag{4.5}$$

for every $f \in C_b^2$.

Accordingly the relaxed cost functional is defined by

$$J(u) = E\left[\int_0^T \int_{\mathbb{A}} h(t, X_t, \mathbb{P}_{X_t}, a)\mu_t(da)dt + g(X_T, \mathbb{P}_{X_T})\right]$$

Remark 4.2 *The set $\mathcal{U}_{ad}$ of strict controls is embedded into the set of relaxed controls by identifying u_t with $dt\delta_{u_t}(da)$.*

It was proved in [29] for classical control problems and in [7] for mean-field control problems, that the relaxed state process corresponding to a relaxed control must satisfies a MFSDE driven by a martingale measure instead of a Brownian motion. That is the relaxed state process satisfies

$$\begin{cases} dX_t = \int_{\mathbb{A}} b(t, X_t, \mathbb{P}_{X_t}, a)\, \mu_t(da)dt + \int_{\mathbb{A}} \sigma(t, X_t, \mathbb{P}_{X_t}, a)\, M(da, dt) \\ X_0 = x, \end{cases} \tag{4.6}$$

where M is an orthogonal continuous martingale measure, with intensity $dt\mu_t(da)$.

The relaxed control problem is therefore defined as follows.

$$\begin{cases} \text{Mimimize } J(\mu) = E\left[\int_0^T \int_{\mathbb{A}} h(t, X_t, \mathbb{P}_{X_t}, u_t)\mu_t(da)dt + g(X_T, \mathbb{P}_{X_T})\right] \\ \text{subject to } X_t = x + \int_0^t \int_{\mathbb{A}} b\left(s, X_s, \mathbb{P}_{X_s}, a\right) \mu_s(da)ds + \int_0^t \int_{\mathbb{A}} \sigma\left(s, X_s, \mathbb{P}_{X_s}, u_s\right) M(da, ds) \end{cases}$$

over the class of relaxed controls.

4.4 Approximation of the relaxed control problem

In this section we will prove that under pathwise uniqueness of the controlled state equation, the relaxed and strict control problems have the same value function. This means that the infimum over the strict controls is equal to the infimum over relaxed controls. This stability property is based on the approximation of the relaxed controls and relaxed trajectories by sequences of strict controls and controlled states.

Let us recall the so-called chattering lemma [15, 30], whose proof is given for the sake of completeness.

Lemma 4.2 (chattering lemma) *Let μ be a relaxed control. Then there exists a sequence of adapted processes (u^n) with values in $\mathbb{A}$, such that the sequence of random measures $\left(\delta_{u_t^n}(da)\, dt\right)$ converges in $\mathbb{V}$ to $dt.\mu_t(da)$, $P-a.s.$, that is for any f continuous in $[0,T]\times\mathbb{A}$, we have :*

$\lim\limits_{n\to+\infty}\int_0^T f(s,u_s^n)ds = \int_0^T\int_{\mathbb{A}} f(s,a)\mu_t(da)$ *uniformly in* $t\in[0,T]$, $P-a.s.$

Proof. Suppose that $\mu(t,da)$ has continuous sample paths. Let $n\geq 1$, let us divide the interval $[0,T]$ into subintervals (T_i) of the form $[t_i,s_i[$of lengh not exceeding 2^{-n}. Cover $\mathbb{A}$ by finitely many disjoint sets (A_j) such that diameter$(A_j)\leq 2^{-n}$. Choose a point (t_i,a_{ij}) in $T_i\times A_j$ for each i,j,t_i being as before. Let $\lambda_{ij}=\mu(t_i,A_j)$, then $\sum_j\lambda_{ij}=1$. Subdivide each T_i further into disjoint left-closed, right-open intervals T_{ij} of length $\lambda_{ij}\times$ the length of T_i. Let $\varepsilon>0$. Since f is uniformly continuous, then for n large enough we have

$$
\begin{aligned}
|f(t,a)-f(t_i,a_{ij})| &< \varepsilon \quad \text{for } (t,a)\in T_i\times A_j \\
\sup_a |f(t,a)-f(t_i,a)| &< \varepsilon \quad \text{for } t\in T_i
\end{aligned}
$$

Defined the sequence of predictable process $\mu_n(.)$ by $\mu_n(t,da)=\delta_{a_{ij}}(da)$ for $t\in T_{ij}$.Then

$$
\begin{aligned}
&\left|\int_0^T\int_{\mathbb{A}} f(t,a)\mu_n(t,da)dt-\int_0^T\int_{\mathbb{A}} f(t,a)\mu(t,da)dt\right| \\
&=\left|\sum_{i,j}\left(\int_{T_{ij}} f(t,a_{ij})dt-\int_{T_{ij}}\int_{\mathbb{A}} f(t,a)\mu(t,da)dt\right)\right| \\
&\leq 2\varepsilon T+\left|\sum_{i,j}\left(\int_{T_{ij}} f(t,a_{ij})dt-\int_{T_{ij}}\int_{\mathbb{A}} f(t_i,a)\mu(t,da)dt\right)\right|
\end{aligned}
$$

By path-continuity of $u(.)$, we may increase n further if necessary to ensuere that the above is bounded by

$$
\begin{aligned}
&3\varepsilon T + \left|\textstyle\sum_{i,j}\left(\int_{T_{ij}} f(t,s_{ij})dt - \int_{T_{ij}}\int_{\mathbb{A}} f(t_i,s)\mu(t_i,da)dt\right)\right| \\
&\leq 4\varepsilon T + \left|\sum_{i,j}\left(\int_{T_{ij}} f(t,s_{ij})dt - \int_{T_{ij}}\int_{\mathbb{A}} f(t_i,s_{ij})\mu(t_i,da)dt\right)\right| \\
&\leq 4\varepsilon T
\end{aligned}
$$

which achieves the proof. Now if $\mu(t,da)$ does not have continuous sample paths, approximate it by controls which do, e.g. by $\mu^n(.)$ defined for a continuous f by

$$\int_{\mathbb{A}} f d\mu^n(t) = k^{-1}\int_{(t-1/n)\vee 0}^{t}\int_{\mathbb{A}} f d\mu(a)da,$$

where $k = [t-(t-1/n)\vee 0]$.$\boxdot$

Let μ *be a relaxed control and* X_t the relaxed state process which is the solution of the MVSDE corresponding to the relaxed control μ :

$$
\begin{cases}
dX_t = \int_{\mathbb{A}}\sigma\left(t,X_t,\mathbb{P}_{X_t},a\right)M(da,dt) + \int_{\mathbb{A}} b\left(t,X_t,\mathbb{P}_{X_t},a\right)\mu_t(da)dt \\
X_0 = x.
\end{cases}
\tag{4.7}
$$

The main result of this section is based on the following stability theorem for martingale measures proved in ([57]).

Theorem 4.4 *([57] page 196) let E be a compact metric space and M be a continous orthogonal martingale measure with covariance $\mu_t(da)dt$ on $[0,T]\times\mathbb{A}$, where μ is a probability measure-valued process. Then there exist on an extension of the initial probability space a sequence of $\mathbb{A}$-valued processes (u^n) and a Brownian motion W such that for each continuous bounded $\varphi:\mathbb{A}\longrightarrow\mathbb{R}$ and $\forall t\in[0,T]$:*
$E\left[\left(\int_0^t\varphi(u_t^n)dt - \int_0^t\int_{\mathbb{A}}\varphi(a)M(dt,da)\right)^2\right]\to 0$

Let (u^n) be the sequence of strict controls approximating μ as in the last lemma and X_t^n be the solution of the state equation(4.1) corresponding to u^n. If we denote $M^n(t,F) =$

$\int_0^t \int_F \delta_{u_s^n}(da)dW_s$, then $M^n(t,F)$ is an orthogonal martingale measure and X_t^n may be written in a relaxed form as follows

$$\begin{cases} dX_t^n = \int_{\mathbb{A}} b(t,X_t^n,\mathbb{P}_{X_t^n},a)\delta_{u_t^n}(da)dt + \int_{\mathbb{A}} \sigma(t,X_t,\mathbb{P}_{X_t^n},a)M^n(dt,da) \\ X_0 = x \end{cases}$$

Therefore X_t^n may be viewed as the solution of (4.6) corresponding to the relaxed control $\mu^n = dt\delta_{u_t^n}(da)$ and the corresponding cost is defined by

$$J(\mu^n) = E\left[\int_0^T \int_{\mathbb{A}} h(t,X_t^n,\mathbb{P}_{X_t^n},u_t)\mu_t^n(da)dt + g(X_T^n,\mathbb{P}_{X_T^n})\right] = J(u^n)$$

Since $\left(\delta_{u_t^n}(da)\,dt\right)$ *converges weakly to* $\mu_t(da)\,dt$, $P-a.s.$, then for every bounded continuous $\varphi : \mathbb{A} \to \mathbb{R}$, we have

$$E\left[\left(\int_0^T \int_{\mathbb{A}} \varphi(a)M^n(dt,da) - \int_0^T \int_{\mathbb{A}} \varphi(a)M(dt,da)\right)^2\right] \to 0 \ as \ n \longrightarrow +\infty. \tag{4.8}$$

Now we are ready to state the main result of this section.

Theorem 4.5 *Let X be the relaxed state process solution of (4.6) and X^n the solution of (4.1) corresponding to u^n. Then if pathwise uniqueness holds for The MVSDE (4.6) we have :*
1) $\lim\limits_{n\to+\infty} E\left[\sup\limits_{t\le 1} |X_t^n - X_t|^2\right] = 0.$
2) If $J(u^n)$ and $J(\mu)$ denote the expected costs corresponding respectively to u^n and μ, then $(J(u^n))$ converges to $J(\mu)$.

The proof is based on the tightness of the processes under consideration.

Lemma 4.3 *The sequence of distributions of the relaxed controls $(\mu^n,\mu)_n$ is tight in $\mathbb{V}$.*

Proof. The relaxed controls μ^n μ are measure valued random variables the space $\mathbb{V}$ which is compact. Then Prohorov's theorem yields that the family of distributions associated to $(\mu^n)_{n\geq 0}$ is tight then it is relatively compact. $\boxdot$

Lemma 4.4 *The family of martingale measures $((M^n)_{n\geq 0}, M)$ is tight on the space $C_{S'} = C([0,1],S')$ of continuous functions from $[0,1]$ with values in S' the topological dual of the Schwartz space S of rapidly decreasing functions.*

Proof. The martingale measures $M_n, n \geq 0, M$ can be considered as random variables with values in $C_{s'} = C([0,T], S')$.

By applying [60], Lemma 6.3, it is sufficient to show that for every f in S, the family $(M_n(\varphi)_{n\geq 0}, M(\varphi))$ is tight in $C = C([0,T], R^d)$ the space of continuous functions endowed with the topology of uniform convergence, where

$$M_n(\varphi) = \int_0^t \int_{\mathbb{A}} \varphi(a) M_n(dt, da) \text{ and } M(\varphi) = \int_0^t \int_{\mathbb{A}} \varphi(a) M(dt, da).$$

Let $p > 1$ and $s < t$. Using Burkholder-Davis-Gundy inequality, we get

$$\begin{aligned} E(|M_t^n(\varphi) - M_s^n(\varphi)|^{2p}) &\leq C_p E[(\int_0^t \int_{\mathbb{A}} |\varphi(a)|^2 \mu_s^n(da) ds)^p] \leq C_p E[(\int_0^t \int_{\mathbb{A}} |\varphi(u_t^n)|^2 dt)^p] \\ &\leq C_p \sup_{a\in\mathbb{A}} |\varphi(a)|^{2p} |t-s|^p \leq K_p |t-s|^p, \end{aligned}$$

where K_p is a constant depending only on p. Therefore the Kolmogorov's criterion is fulfilled. Hence the sequence $(M_n(\varphi))$ is tight in $C([0,T], R^d)$. Similar arguments can be used to show that $E(|M_t(\varphi) - M_s(\varphi)|^{2p}) \leq K_p |t-s|^p$, which yields the tightness of $M_t(\varphi)$. $\boxdot$

Lemma 4.5 *The family (X^n, X) of solutions of MVSDEs corresponding to the controls μ_t^n and μ_t is tight in $C([0,T]; \mathbb{R}^d)$.*

Proof. Let us verify condition i) and ii) of Theorem 2.5. $X^n(0) = X(0) = x$, then it is obvious that a) is satisfied. Moreover let $t > s$, we have

$$\begin{aligned} E\left(|X_t^n - X_s^n|^4\right) &\leq E\left(\left|\int_s^t \int_{\mathbb{A}} b\left(u, X_u^n, \mathbb{P}_{X_u^n}, a\right) \mu_s^n(da).ds\right|^4\right. \\ &\quad \left. + \left|\int_s^t \int_{\mathbb{A}} \sigma\left(u, X_u^n, \mathbb{P}_{X_u^n},\right) dM^n(da, du)\right|^4\right). \end{aligned}$$

Since b and σ are bounded functions, then by using Burkholder-Davis-Gundy inequality to the martingale part yields $E\left(|X_t^n - X_s^n|^4\right) \leq K\left|t-s\right|^2$ The same arguments could be used for X. Therefore the family (X^n, X) is tight in $C([0,T])$. $\boxdot$

Limit theorems for stochastic integrals play a key role in the proof. Let us recall some limit theorems for stochastic integrals due initially to Skorokhod ([65] or [37] Lemma 3.1) for the Brownian motion and then extended to stochastic integrals driven by local martingales by ([36]).

Lemma 4.6 *Consider a family of filtrations* $(F_t^n), (F_t)$ *satisfying the usual conditions. Let* $\{f_n(t), f(t) : t \in [0,T]\}$ *be a sequence of continuous adapted processes and let* $\{N_n(t), N(t) : t \in [0,T]\}$ *be a sequence of continuous local martingales with respect to* $(F_t^n), (F_t)$ *respectively. Suppose that*

$\lim_{n\to+\infty} f_n = f$ *in probability in* $C([0,T])$.

$\lim_{n\to+\infty} N_n = N$ *in probability in* $C([0,T])$.

Then

$$\forall\varepsilon > 0 \ , \qquad \lim_{n\to+\infty} P\left[\sup_{t\leq 1}\left|\int_0^T f_n dN_n - \int_0^T f dN\right| > \varepsilon\right] = 0$$

Proof of Theorem 4.3. Let X, X^n be the solutions of (4.6) corresponding to the controls μ and μ^n. Suppose that the first statement of the theorem is false. Then there exists $\delta > 0$ such that :

$$\inf_n E\left[\sup_{t\leq 1}|X_t^n - X_t|^2\right] \geq \delta. \tag{4.9}$$

It is clear from the last Lemmas that the family of distributions of $(\mu^n, \mu, M^n, M, X^n, X, \mathbb{P}_{X^n}, \mathbb{P}_X)$ is tight on the space $\Lambda = \mathbb{V}^2 \times C_{S'}^2 \times C^2 \times \mathcal{P}(C)^2$. Then by Prokhorov's Theorem it is relatively compact with respect to weak topology. Therefore by Skorokhod's selection Theorem there exist a probability space $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P}\right)$ and a sequence $\left(\widetilde{\Gamma}^n\right) = (\widetilde{\mu}^n, \widetilde{\nu}^n, \widetilde{M}^n, \widetilde{N}^n, \widetilde{X}^n, \widetilde{Y}^n)$ such that :

A) For each $n \in \mathbf{N}$, the laws of $(\mu^n, \mu, M^n, M, X^n, X)$ and $(\widetilde{\mu}^n, \widetilde{\nu}^n, \widetilde{M}^n, \widetilde{N}^n, \widetilde{X}^n, \widetilde{Y}^n)$ coincide.

B) There exists a subsequence still denoted by $\left(\widetilde{\Gamma}^n\right)$ (to avoid heavy notations) which converges to $\widetilde{\Gamma} = (\widetilde{\mu}, \widetilde{\nu}, \widetilde{M}, \widetilde{N}, \widetilde{X}, \widetilde{Y})$, $\widetilde{P}-$ $a.s$ on the space Λ.

Let $\widetilde{\mathcal{F}_t^n} = \sigma\left(\widetilde{\mu}_s^n, \widetilde{\nu_s}^n, \widetilde{M_s^n}, \widetilde{N_s}^n, \widetilde{X_s}^n, \widetilde{Y_s}^n; s \leq t\right)$ and $\widetilde{\mathcal{F}_t} = \sigma\left(\widetilde{\mu}_s, \widetilde{\nu}_s, \widetilde{M}_s, \widetilde{N}_s, \widetilde{X}_s, \widetilde{Y}_s; s \leq t\right)$

It is clear that $\widetilde{M_s^n}, \widetilde{N_s}^n$ and $\widetilde{M}_s, \widetilde{N}_s$ are continuous orthogonal martingale measures with respect to $\left(\widetilde{\mathcal{F}_t^n}\right)$ and $\left(\widetilde{\mathcal{F}_t}\right)$

Property A) implies that $\widetilde{X^n}$and $\widetilde{Y^n}$ satisfy the following equations :

$$\begin{cases} d\widetilde{X_t^n} = \int_{\mathbb{A}} \sigma\left(t, \widetilde{X_t^n}, \widetilde{P}_{\widetilde{X_t^n}}, a\right) \widetilde{M}^n(da, dt) + \int_{\mathbb{A}} b\left(t, \widetilde{X_t}^n, \widetilde{P}^n_{\widetilde{X_t^n}}, a\right) \widetilde{\mu}^n(t, da)\, dt \\ X_0'^n = x \end{cases} \tag{4.10}$$

$$\begin{cases} d\widetilde{Y_t^n} = \int_{\mathbb{A}} \sigma\left(t, \widetilde{Y_t^n}, \widetilde{P}_{\widetilde{Y_t^n}}, a\right) \widetilde{N}^n(da, dt) + \int_{\mathbb{A}} b\left(t, \widetilde{Y_t^n}, \widetilde{P}_{\widetilde{Y_t^n}}, a\right) \widetilde{\nu}^n(t, da)\, dt \\ Y_0'^n = x. \end{cases} \tag{4.11}$$

By letting n going to infinity in equations 4.10 and 4.11 and using Skorokhod's limit theorem (Lemma 4.7) and property B), it turns out that the processes $\widetilde{X}$ and $\widetilde{Y}$ satisfy the following equations :

$$\begin{cases} d\widetilde{X_t} = \int_{\mathbb{A}} \sigma\left(t, \widetilde{X_t}, \widetilde{P}_{\widetilde{X_t}}, a\right) \widetilde{M}(da, dt) + \int_{\mathbb{A}} b\left(t, \widetilde{X_t}, \widetilde{P}_{\widetilde{X_t}}, a\right) \widetilde{\mu}(t, da)\, dt \\ \widetilde{X}_0 = x \end{cases} \tag{4.12}$$

$$\begin{cases} d\widetilde{Y}_t = \int_{\mathbb{A}} \sigma\left(t, \widetilde{Y}_t, \widetilde{P}_{\widetilde{Y}_t}, a\right) \widetilde{N}(da, dt) + \int_{\mathbb{A}} b\left(t, \widetilde{Y}_t, \widetilde{P}_{\widetilde{Y}_t}, a\right) \widetilde{\nu}(t, da)\, dt \\ \widetilde{Y}_0 = x. \end{cases} \tag{4.13}$$

We have by the chattering lemma that $\mu^n \to \mu$ in $\mathbb{V}$, P $a.s$, then the sequence (μ^n, μ) converges to (μ, μ) in $\mathbb{V}^2$. Moreover $law\,(\mu^n, \mu) = law(\widetilde{\mu}^n, \widetilde{\nu}^n)$ and $(\widetilde{\mu}^n, \widetilde{\nu}^n) \longrightarrow (\widetilde{\mu}, \widetilde{\nu})$, $\widetilde{P}$ $-a.s$ in $\mathbb{V}^2$.

Therefore $\text{law}(\widetilde{\mu}, \widetilde{\nu}) = \text{law}(\mu, \mu)$, which is supported by the diagonal of $\mathbb{V}^2$. Therefore $\widetilde{\mu} = \widetilde{\nu}$, $\widetilde{P} - a.s.$

Using the same arguments, we show that $\widetilde{M}(da, dt) = \widetilde{N}(da, dt), \widetilde{P} - a.s.$

It follows that $\widetilde{X}$ and $\widetilde{Y}$ are solutions of the same stochastic differential equation driven by the same matingale measure $\widetilde{M}(da, dt)$ and the same relaxed control $\widetilde{\mu}(t, da)\, dt$. Therefore since $\widetilde{X}(0) = \widetilde{Y}(0)$, then by pathwise uniquenes we have $\widetilde{X} = \widetilde{Y}$, $\widetilde{P}-$ $a.s.$

Using properies A) and B) we have :

$$\begin{aligned}\delta &\leq \liminf_{n\in\mathbf{N}} E\left[\sup_{t\leq T}\left|X_t^n - X_t\right|^2\right] = \liminf_{n\in\mathbf{N}} \widetilde{E}\left[\sup_{t\leq T}\left|\widetilde{X_t^n} - \widetilde{Y_t^n}\right|^2\right] \\ &= E'\left[\sup_{t\leq T}\left|\widetilde{X}_t - \widetilde{Y}_t\right|^2\right] = 0\end{aligned}$$

which contradicts (4.9).

2)Let u^n and μ as in 1) then

$$\begin{aligned}\left|J\left(u^n\right) - J\left(\mu\right)\right| &\leq E\left[\int_0^T \int_{\mathbb{A}} \left|h(t, X_t^n, \mathbb{P}_{X_t^n}, a) - h(t, X_t, \mathbb{P}_{X_t}, a)\right| \delta_{u_t^n}(da)\, dt\right] \\ &+ E\left[\left|\int_0^T \int_{\mathbb{A}} h(t, X_t, \mathbb{P}_{X_t}, a)\delta_{u_t^n}(da)\, dt - \int_0^T \int_{\mathbb{A}} h(t, X_t, \mathbb{P}_{X_t}, a)\mu_t(da)\, dt\right|\right] \\ &+ E\left[\left|g(X_T^n, \mathbb{P}_{X_T^n}) - g(X_T, \mathbb{P}_{X_T})\right|\right]\end{aligned}$$

The first assertion implies that the sequence (X_t^n) converges to X_t in probability. It can be easily verified that 2) follows from the continuity and boundedness of the coefficients and an application of the dominated convergence theorem.$\boxdot$

Remark 4.3 *a) According to the last Proposition, it is clear that the infimum among relaxed controls is equal to the infimum among strict controls. This implies that the value functions for the relaxed and strict models are the same.*

*b) The result is valid under any assumption on the coefficients ensuring pathwise uniqueness. In particular the result is valid in the case of Lipschitz coefficients or under Osgood type condition [**?**].*

c) The assumption on the boundedness of the coefficients could be replaced by a linear growth condition.

4.5 Existence of optimal controls

4.5.1 Existence of optimal relaxed controls

In this section we use similar techniques to prove the existence of an optimal relaxed control.

Theorem 4.6 *Assume ($\boldsymbol{H_1}$), then the relaxed control problem admits an optimal solution.*

Proof. Let $\delta = \inf \{J(\mu); \mu \in \mathcal{R}\}$ where

$$J(u) = E\left[\int_0^T \int_{\mathbb{A}} h(t, X_t, \mathbb{P}_{X_t}, a)\mu_t(da)dt + g(X_T, \mathbb{P}_{X_T})\right]$$

Let $(\mu^n, X_n)_{n\geq 0}$ be a minimizing sequence for the cost function $J(\mu)$, that is, $\lim\limits_{n\to\infty} J(\mu^n) = \delta$, where X^n is the solution of

$$\begin{cases} dX_t^n = \int_{\mathbb{A}} \sigma\left(t, X_t^n, \mathbb{P}_{X_t^n}, a\right) M^n(da, dt) + \int_{\mathbb{A}} b\left(t, X_t^n, \mathbb{P}_{X_t^n}, a\right) \mu_t^n(da)dt \\ X_0^n = x. \end{cases}$$

The coefficients of the dynamics are bounded, then using similar arguments as in the last section we show easily that the family of processes $(\mu^n, M^n, X^n, \mathbb{P}_{X^n})$ is tight in the space $\Lambda = \mathbb{V} \times C_{S'} \times C^2 \times \mathcal{P}(C)$. Applying Skorokhod's selection Theorem, there exist a probability space $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P}\right)$ and a sequence $\left(\widetilde{\Gamma}^n\right) = (\widetilde{\mu}^n, \widetilde{M}^n, \widetilde{X}^n)$ such that :

i) For each $n \in \mathbf{N}$, the laws of (μ^n, M^n, X^n) and $(\widetilde{\mu}^n, \widetilde{M}^n, \widetilde{X}^n)$ coincide.

ii) There exists a subsequence still denoted by $\left(\widetilde{\Gamma}^n\right)$ which converges to $\widetilde{\Gamma} = (\widetilde{\mu}, \widetilde{M}, \widetilde{X})$, $\widetilde{P}-$ $a.s$ on the space Λ.

Let $\widetilde{\mathcal{F}_t^n} = \sigma\left(\widetilde{\mu}_s^n, \widetilde{M}_s^n, \widetilde{X}_s^n; s \leq t\right)$ and $\widetilde{\mathcal{F}}_t = \sigma\left(\widetilde{\mu}_s, \widetilde{M}_s, \widetilde{X}_s; s \leq t\right)$

$\widetilde{M}_s^n, \widetilde{N}_s^n$ and $\widetilde{M}_s, \widetilde{N}_s$ are continuous orthogonal martingale measures with respect to $\left(\widetilde{\mathcal{F}_t^n}\right)$ and $\left(\widetilde{\mathcal{F}}_t\right)$

Property i) implies that $\widetilde{X^n}$ satisfies the following MVSDE :

$$\begin{cases} d\widetilde{X_t^n} = \int_{\mathbb{A}} \sigma\left(t, \widetilde{X_t^n}, \widetilde{P}_{\widetilde{X_t^n}}, a\right) \widetilde{M}^n(da, dt) + \int_{\mathbb{A}} b\left(t, \widetilde{X}_t^n, \widetilde{P}_{\widetilde{X_t^n}}^n, a\right) \widetilde{\mu}^n(t, da)\, dt \\ X_0'^n = x \end{cases}$$

By letting n going to infinity, using Skorokhod's limit theorem and according to ii) we see that the processes $\widetilde{X}$ satisfy the MVSDE :

$$\begin{cases} d\widetilde{X_t} = \int_{\mathbb{A}} \sigma\left(t, \widetilde{X_t}, \widetilde{P}_{\widetilde{X_t}}, a\right) \widetilde{M}(da, dt) + \int_{\mathbb{A}} b\left(t, \widetilde{X_t}, \widetilde{P}_{\widetilde{X_t}}, a\right) \widetilde{\mu}\,(t, da)\,dt \\ \widetilde{X}_0 = x \end{cases}$$

The instantaneous cost h and the final cost g are continuous and bounded in (x, a), we proceed as in Corollary 3.2, to conclude that $\delta = \lim_{n\to\infty} J(\mu^n) = J(\widetilde{\mu})$. Hence $\widetilde{\mu}$ is an optimal relaxed control.$\boxdot$

4.5.2 Existence of strict optimal controls

In this section we show that under convexity condition on the coefficients, the relaxed optimal control is in fact realized by a strict optimal control. This fact is based on some measurable selection theorem and Caratheodory Lemma. We start by proving that since the action space is compact, we can restrict the investigation for an optimal relaxed control to the smaller class of so-called *sliding controls* also known as *chattering controls*, having the form

$$\nu_t = \textstyle\sum_{i=1}^{p} \alpha_i(t) dt \delta_{u_i(t)}(da),\ u_i(t) \in \mathbb{A}, \alpha_i(t) \geq 0 \text{ and } \sum_{i=1}^{p} \alpha_i(t) = 1. \tag{4.14}$$

where $\alpha_i(t)$ and $u_i(t)$ are adapted stochastic processes.

Note that if ν_t has the form (4.14) then the relaxed controlled state process is solution of

$$\begin{cases} dX_t &= \sum_{i=1}^{p} \alpha_i(t) b(t, X_t, \mathbb{P}_{X_t}, u_i(t))dt + \sum_{i=1}^{p} \alpha_i(t)^{1/2} \sigma(t, X_t, \mathbb{P}_{X_t}, u_i(t)) dW_t^i \\ X_0 &= x \end{cases} \tag{4.15}$$

Proposition 4.1 *Let μ be a relaxed control and X^{μ} the corresponding state process solution of (4.6). Then there exists a sliding control*

$$\nu_t = \textstyle\sum_{i=1}^{p}\alpha_i(t)\delta_{u_i(t)}(da),\ \ u_i(t) \in A,\ \ \alpha_i(t) \geq 0 \text{ and } \textstyle\sum_{i=1}^{p}\alpha_i(t) = 1 \tag{4.16}$$

such that :

1) $X^{\mu} = X^{\nu}$.

2) $J(\mu) = J(\nu)$.

Proof.

Let Λ denote the N-dimensional simplex where $N = d + d^2 + 1$

$$\Lambda = \left\{\lambda = (\lambda_0, \lambda_1, ..., \lambda_N)\,;\ \lambda_i \geq 0;\ \textstyle\sum_{i=1}^{N}\lambda_i = 1\right\}$$

and W the $(N+1)$-cartesian product of the set $\mathbb{A}$

$$W = \{w = (u_0, u_1, ..., u_N)\,;\ u_i \in \mathbb{A}\}$$

Define the function

$$g(t, \lambda, w) = \textstyle\sum_{i=1}^{N}\lambda_i\widetilde{b}(t, X_t, \mathbb{P}_{X_t}, u_i) - \int_{\mathbb{A}}\widetilde{b}\,(t, X_t, \mathbb{P}_{X_t}, a)\,\mu_t(da)$$

where $t \in [0, T]\,, \lambda \in \Lambda, w \in W$ and $\widetilde{b}(t, X_t, \mathbb{P}_{X_t}, u_i) = \begin{pmatrix} b(t, X_t, \mathbb{P}_{X_t}, u_i) \\ \sigma\sigma^*(t, X_t, \mathbb{P}_{X_t}, u_i) \\ h(t, x_t, \mathbb{P}_{X_t}, u_i) \end{pmatrix}$

Let $\widetilde{b}(t, X_t, \mathbb{P}_{X_t}, u_i)$, $i = 0, 1, ..., N$, be arbitrary points in $P(t, X_t)$ where

$$\mathcal{V}(t, X_t) = \{(b(t, X_t, \mathbb{P}_{X_t}, a),\ \sigma\sigma^*(t, X_t, \mathbb{P}_{X_t}, a),\ h(t, X_t, \mathbb{P}_{X_t}, a))\,;\ a \in \mathbb{A}\} \subset \mathbb{R}^N$$

Then the convex hull of this set is the collection of all points of the form $\sum_{i=1}^{N}\lambda_i\widetilde{b}(t, X_t, \mathbb{P}_{X_t}, u_i)$. If μ is a relaxed control, then $\int_{\mathbb{A}}\widetilde{b}\,(t, X_t, \mathbb{P}_{X_t}, a)\,\mu_t(da) \in Conv\,(\mathcal{V}(t, X_t))$, the convex hull of $\mathcal{V}(t, X_t)$. Therefore it follows from Carathéodory's Lemma (which says that the convex hull of a d-dimensional set M coincides with the union of the convex hulls of $d + 1$ points of M), that for each $(\omega, t) \in \Omega \times [0, T]$ the equation $g(t, \lambda, \omega) = 0$ admits at least one solution. Moreover the set

$$\left\{(\omega,\lambda,w)\in\Omega\times\Lambda\times W:\textstyle\sum_{i=1}^{N}\lambda_i\widetilde{b}(t,X_t,\mathbb{P}_{X_t},u_i)=\int_{\mathbb{A}}\widetilde{b}\,(t,x_t,\mathbb{P}_{X_t},a)\,\mu_t(da)\right\}$$

is measurable with respect to $\mathcal{F}_t\otimes\mathcal{B}(\mathbb{R}^{d+1})\otimes\mathcal{B}(\mathbb{A}^{d+1})$ with non empty $\omega-$sections for each ω.

Hence by using a measurable selection theorem [30], there exist measurable $\mathcal{F}_t-$adapted processes λ_t and w_t with values, respectively in Λ and W such that :

$$\textstyle\int_{\mathbb{A}}\widetilde{b}\,(t,X_t,\mathbb{P}_{X_t},a)\,\mu_t(du)=\sum_{i=1}^{N}\lambda_i(t)\widetilde{b}(t,X_t,\mathbb{P}_{X_t},u_i(t))$$

Then it is easy to verify that the process defined by its drift $\sum_{i=1}^{N}\lambda_i(t)b(t,X_t,\mathbb{P}_{X_t},u_i(t))$ and its quadratic variation $\sum\limits_{i=0}^{N}\lambda_i(t)\sigma\sigma^*(t,X_t,\mathbb{P}_{X_t},u_i(t))$ is the solution of the MVSDE (4.15), defined possibly on an extension of the initial probability space because of the possible degeneracy of the matrix $\sigma\sigma^*$. $\boxdot$

Corollary 4.1 *Suppose that the following set*

$$P(t,X_t)=\left\{\widetilde{b}(t,X_t,\mathbb{P}_{X_t},a);\ a\in\mathbb{A}\right\}\subset\mathbb{R}^N \tag{4.17}$$

is convex, where $\widetilde{b}(t,X_t,\mathbb{P}_{X_t},a)=(b(t,X_t,\mathbb{P}_{X_t},a),\ \sigma\sigma^*(t,X_t,\mathbb{P}_{X_t},a),\ h(t,X_t,\mathbb{P}_{X_t},a))$.
Then the strict control problem admits an optimal solution.

Proof.

Let μ be an optimal relaxed control whose existence is garanteed by Theorem 4.5. Then by using Proposition 5.1, it follows that for each relaxed control μ we have

$$\textstyle\int_{\mathbb{A}}\widetilde{b}\,(t,X_t,\mathbb{P}_{X_t},a)\,\mu_t(da)\in Conv\,(\mathcal{V}(t,X_t))$$

Since $\mathcal{V}(t,X_t)$ is convex then $Conv\,(\mathcal{V}(t,X_t))=\mathcal{V}(t,X_t)$. Then applying the same arguments as in Proposition 5.1, there exists a measurable $\mathcal{F}_t-$adapted process u_t such that

$$\textstyle\int_{\mathbb{A}}\widetilde{b}\,(t,X_t,\mathbb{P}_{X_t},a)\,\mu_t(du)=\widetilde{b}(t,X_t,,\mathbb{P}_{X_t},u_t).$$

This implies that X_t is a solution of the MVSDE

$$\begin{cases} dX_t = b(t, X_t, , \mathbb{P}_{X_t}, u_t)dt + \sigma(t, X_t, , \mathbb{P}_{X_t}, u_t)dW_t \\ X_0 = x \end{cases}$$

and $J(\mu) = J(u)$. $\boxdot$

Remark 4.4 *The convexity condition (4.17) may be replaced by the following condition* $\left\{ \left(\overline{b}(t, X_t, \mathbb{P}_{X_t}, a), z\right) ;\ a \in \mathbb{A},\ h(t, X_t, \mathbb{P}_{X_t}, a) \geq z \right\} \subset \mathbb{R}^{d+d^2} \times \mathbb{R}$ *is convex, where* $\overline{b}(t, X_t, \mathbb{P}_{X_t}, a) = (b(t, X_t, \mathbb{P}_{X_t}, a), \sigma\sigma^*(t, X_t, \mathbb{P}_{X_t}, a))$.

4.5.3 Singular optimal control problems

We consider the mixed control problem where the state process is governed by a MVSDE allowing both classical and singular control. Without loss of generality and to avoid heavy notations with indices, we suppose that all our processes are one dimensional. More precisely the state evolves according to the MVSDE

$$\begin{cases} dX_t = b\left(t, X_t, \mathbb{P}_{X_t}, u_t\right) dt + \sigma\left(t, X_t, \mathbb{P}_{X_t}, u_t\right) dB_t + g(t) dv_t \\ X_0 = x. \end{cases} \tag{4.18}$$

The cost functional is defined by

$$J(u, v) = E\left[\int_0^T h(t, X_t, \mathbb{P}_{X_t}, u_t)dt + \int_0^T c(t) dv_t\right]$$

where in addition of assumptions $(\mathbf{H_1})$, we suppose the following.

$(\mathbf{H_2})$ the singular control v is a progressively measurable process with values in the space $\mathcal{A}([0, T])$: the set of $\mathbb{R}$-valued continuous increasing functions that are left continuous and have right limits such that $v_0 = 0$. Let us denote the set $\mathcal{V}_{ad}$ of such singular controls.

$g : [0, T] \longrightarrow \mathbb{R}$ is a bounded continuous function.

$c : [0, T] \longrightarrow \mathbb{R}$ is a continuous increasing function such that $c(t) > 0$.

By analogy to the classical stochastic controls problems (see [40]), we have a similar definition of a relaxed control for MVSDEs .

Definition 4.7 *A relaxed singular control is the term* $\alpha = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P, \mu, X, v, x)$ *such that*

1) $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ *is a probability space equipped with a filtration* $(\mathcal{F}_t)_{t\geq 0}$ *satisfying the usual conditions ;*

2) μ *is measure valued control, progressively measurable with respect to* $(\mathcal{F}_t)$ *;*

3) v *is a progressively measurable process with values in the space* $\mathcal{A}([0,T])$.

4) (X_t) *is* $\mathbb{R}^d$*-valued,* $\mathcal{F}_t$*-adapted, with continuous paths, such that*

$$f(X_t) - \int_0^t \int_A L^{P_{X_s}} f(s, X_s, a)\, \mu_s(da) ds - \int_0^t \nabla_x f(X_s) g(s) dv_s$$
$$- \sum_{0\leq s\leq t} [f(X_{s+}) - f(X_s) - \nabla_x f(X_s) \Delta X_s] \text{ is a } P - \text{martingale}, \tag{4.19}$$

for every $f \in C_b^2$.

Using the same arguments as in the last section, the pathwise representation of the solution of the martingale problem is in fact solution of the MVSDE

$$\begin{cases} dX_t = \int_{\mathbb{A}} b(t, X_t, \mathbb{P}_{X_t}, a)\, \mu_t(da) dt + \int_{\mathbb{A}} \sigma(t, X_t, \mathbb{P}_{X_t}, a)\, M(da, dt) + g(t) dv_t \\ X_0 = x, \end{cases} \tag{4.20}$$

where M is an orthogonal continuous martingale measure, with intensity $dt\mu_t(da)$.
The relaxed cost functional is defined by :

$$J(\mu, v) = E\left[\int_0^T \int_{\mathbb{A}} h(t, X_t, \mathbb{P}_{X_t}, u_t) dt + \int_0^T c(t) dv_t\right]$$

Theorem 4.7 *1) Assume (**H₁**) and (**H₂**). Then the mixed relaxed singular control problem admits an optimal solution.*

2) Suppose that the set

$$P(t, X_t) = \left\{\widetilde{b}(t, X_t, \mathbb{P}_{X_t}, a);\ a \in \mathbb{A}\right\} \subset \mathbb{R}^3 \tag{4.21}$$

is convex, where $\widetilde{b}(t, X_t, \mathbb{P}_{X_t}, a) = (b(t, X_t, \mathbb{P}_{X_t}, a),\ \sigma^2(t, X_t, \mathbb{P}_{X_t}, a),\ h(t, X_t, \mathbb{P}_{X_t}, a))$.

Then the strict control problem admits an optimal solution.

Proof. 1) The proof combines Theorem 4.4 and [40] Theorem 3.8. In fact the difference with Theorem 4.4 is the additional singular control, which can be identified as a random variable with values in the set of increasing functions. The the set of increasing functions is then identified with a closed subset of the set of signed Radon measures on $[0, T]$ for which tightness results are easy to obtain. See [40] for the complete analysis of this part.
2) Use the same arguments as in the classical (without the singular part) control problem. ⊡

4.6 Conclusion

In this paper, we have investigated existence of optimal controls for systems driven by general non linear McKean-Vlasov stochastic differential equations. We have proved under minimal assumptions on the coefficients, the existence of a relaxed optimal control, which is a measure valued control. Moreover, we have showed that relaxing the initial strict control problem does not affect the value function of the strict control problem. In case we have some additional convexity hypothesis, the relaxed optimal control is realized by a strict control. Note that our results could be extended to unbounded coefficients, provided some linear growth conditions must be imposed. The case of measurable coefficients could be handled by using Krylov estimates for stochastic integrals, provided that the matrix $\sigma\sigma^*$ is uniformly elliptic. Optimal control of MVSDEs with jumps, driven by a Brownian motion and an independent Poisson random measure will be subject of our future work.

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Equations differentielles stochastique du type McKean-Vlasov et leur contrôle optimal

Nous considérons les équations différentielles stochastiques (EDS) de Mc Kean-Vlasov, qui sont des EDS dont les coefficients de dérive et de diffusion dépendent non seulement de l'état du processus inconnu, mais également de sa loi de probabilité. Ces EDS, également appelées EDS à champ moyen, ont d'abord été étudiées en physique statistique et représentent en quelque sorte le comportement moyen d'un nombre infini de particules. Récemment, ce type d'équations a suscité un regain d'intérêt dans le contexte de la théorie des jeux à champ moyen. Cette théorie a été inventée par P.L. Lions et J.M. Lasry en 2006, pour résoudre le problème de l'existence d'un équilibre de Nash approximatif pour les jeux différentiels, avec un grand nombre de joueurs. Ces équations ont trouvé des applications dans divers domaines tels que la théorie des jeux, la finance mathématique, les réseaux de communication et la gestion des ressources pétrolières. Dans cette thèse, nous avons étudié les questions de stabilité par rapport aux données initiales, aux coefficients et aux processus directeurs des équations de McKean-Vlasov. Les propriétés génériques de ce type d' équations stochastiques, telles que l'existence et l'unicité, la stabilité par rapport aux paramètres, ont été examinées. En théorie du contrôle, notre attention s'est portée sur l'existence et l'approximation de contrôles relaxés pour les systèmes gouvernés par des EDS de Mc Kean-Vlasov.

Mot clés : EDS de Mc Kean-Vlasov, EDS de type champ moyen, Stabilité, Approximation, Propriété générique, Contrôle relaxé.

McKean-Vlasov stochastic differential equations and their optimal control

We consider Mc Kean-Vlasov stochastic differential equations (SDEs), which are SDEs where the drift and diffusion coefficients depend not only on the state of the unknown process but also on its probability distribution. These SDEs called also mean- field SDEs were first studied in statistical physics and represent in some sense the average behavior of an infinite number of particles. Recently there has been a renewed interest for this kind of equations in the context of mean-field game theory. Since the pioneering papers by P.L. Lions and J.M. Lasry, mean-field games and mean-field control theory has raised a lot of interest, motivated by applications to various fields such as game theory, mathematical finance, communications networks and management of oil resources. In this thesis, we studied questions of stability with respect to initial data, coefficients and driving processes of Mc Kean-Vlasov equations. Generic properties for this type of SDEs, such as existence and uniqueness, stability with respect to parameters, have been investigated. In control theory, our attention were focused on existence, approximation of relaxed controls for controlled Mc Kean-Vlasov SDEs.

Keywords : McKean-Vlasov SDE, mean-field SDE, stability, approximation, generic property, relaxed control.

نعتبر المعادلات التفاضلية العشوائية ، حيث لا تعتمد معاملات الانجراف والانتشار على الحالة غير المعروفة فحسب ، بل تعتمد أيضًا على توزيع الاحتمالات. تمت دراسة هذه SDEs التي تسمى أيضًا SDEs ذات المجال المتوسط لأول مرة في الفيزياء الإحصائية وتمثل بشكل ما متوسط سلوك عدد لا حصر له من الجسيمات. في الآونة الأخيرة ، كان هناك اهتمام متجدد بهذا النوع من المعادلات في سياق نظرية لعبة المجال المتوسط. منذ الأوراق الرائدة من قبل PL Lions و JM Lasry ، أثارت الألعاب الميدانية المتوسطة ونظرية التحكم في المجال المتوسط الكثير من الاهتمام ، مدفوعة بالتطبيقات في مجالات مختلفة مثل نظرية الألعاب والتمويل الرياضي وشبكات الاتصالات وإدارة الموارد النفطية. في هذه الأطروحة ، درسنا أسئلة الاستقرار فيما يتعلق بالبيانات الأولية والمعاملات والعمليات الدافعة لمعادلات Mc Kean-Vlasov. تم التحقق من الخصائص العامة لهذا النوع من SDEs ، مثل الوجود والتفرد ، والاستقرار فيما يتعلق بالمعاملات. في نظرية التحكم ، انصب اهتمامنا على الوجود ، وتقريب عناصر التحكم المريحة من أجل وحدات التحكم SDEs الخاضعة للرقابة Mc Kean-Vlasov.

الكلمات المفتاحية: معادلات McKean-Vlasov SDE ، متوسط المجال SDE ، الاستقرار ، التقريب ، الخاصية العامة ، التحكم المريح.